$$c_C = (1-\pi^{pa}(0)) \times L_{q|M/M/1/(\gamma+1)} \times h_p + S \times \pi^p(0) \times \lambda_d \,.$$

where $\pi^{pa}(0)$ and $\pi^{p}(0)$ are calculated according to Eq. A.2 and Eq. A.4 respectively (see Appendix).

Proof: Similar to lemma 2 for calculating c_B, all possible states can be divided into two partitions, based on the fact that part storage is empty, with the probability of $\pi^{pa}(0)$, or not empty, with the probability of 1 - $\pi^{pa}(0)$. Completing the steps lead to lemma 3.

Lemma 4: c_D can be written as follows:

$$c_D = D \times (\lambda_{dis} + \pi^m(\alpha) \times \lambda_m + \pi^{pa}(\beta) \times \lambda_{pa}) \,.$$

where $\pi^{m}(\alpha)$ and $\pi^{pa}(\beta)$ are calculated according to lemma 1 and Eq. A.3, respectively.

Proof: Since the mean time for decomposing returned items is assumed to be negligible, cost of block *D* only includes disposal costs. Therefore, there is no inventory on hand in this block. As mentioned in section 2, the returned items can be disposed under three circumstances: first, when the returned items are not recoverable and they will be disposed with the rate of λ_{dis}; second, when the material storage is full with probability of $\pi^m(\alpha)$, the recovered item is disposed with the rate of $\pi^m(\alpha) \times \lambda_m$, and thirdly, when the part storage is full with probability of $\pi^{pa}(\beta)$, the recovered item is disposed with the rate of $\pi^{pa}(\beta) \times \lambda_{pa}$. Therefore, the expected total cost of this block can be calculated by multiplying the summation of these three disposal rates by disposal cost per unit (*D*).

Lemma 5: The total cost of the remanufacturing system under proposed $(Q, \alpha, \beta, \gamma)$ inventory control policy can be expressed as the summation of the related costs of four blocks and is formulated as follows:

$$Z = E(x) \times h_m + \pi^p(\gamma) \times \beta \times h_{pa} + \left(1-\pi^p(\gamma)\right) \times L_{q|M/M/1/(\beta+1)} \times h_{pa} + \left(1-\pi^{pa}(0)\right) \times L_{q|M/M/1/(\gamma+1)} \times h_p + S \times \pi^p(0) \times \lambda_d + D \times (\lambda_{dis} + \pi^m(\alpha) \times \lambda_m + \pi^{pa}(\beta) \times \lambda_{pa}) \,.$$

Proof: Putting all cost function for blocks we have calculated we obtained the total cost function above.

4. Solution Procedure

In this section, we develop a Simulated Annealing (SA) based heuristic to obtain near optimal parameters of inventory control policy of proposed cost function. A solution of our problem in the proposed SA is a vector consisted of inventory control policy parameters, namely α, β, γ, and Q. The scheme of proposed SA algorithm to find values of inventory control policy parameters for our remanufacturing problem is depicted in Table 1. In steps 1 and 2, the initial solution and initial temperature are determined. Then, the new solutions are created at each temperature with the defined neighborhood function explained in step 5. Number of new solutions in each temperature is equal to the number of iterations defined by $T/(1+0.2*T)$ (for more details see Lundy and Mees, 1986). If the new solution satisfies the conditions of step 7, it is considered as the input of neighborhood function in step 5-a. Furthermore, according to step 9, the temperature decreases with the rate *c* and when the condition in step 10 is satisfied, the algorithm terminates.

5. Computational Results

In subsection 5.1, validation of proposed estimated cost function is investigated. Then, in subsection 5.2, performance of the proposed SA is evaluated by comparing its results with near optimal values obtained by a Genetic Algorithm (GA) proposed by Roy *et al*. (2008) for a remanufacturing problem.

The real GA is used with following parameters: population size = 30, probability of crossover = 0.2, probability of mutation = 0.1, and maximum number of generation = 300. Similar to our proposed SA, these parameters are selected by trial and error to obtain good performance. The SA procedure was run with the following set of parameters: initial temperature: 0.5, cooling schedule rule: exponential, $T_{K+1} = 0.9T_K$, the values of decreasing and increasing: 1, initial inventory policy parameters are values considered to be $\alpha = \beta = \gamma = Q = 1$. The values of the mentioned parameters and cooling schedule rule are selected by trial and error through experiments to tune the algorithm for a good performance.

To construct scenarios, we consider five parameters, namely, material inventory cost (h_m), part inventory cost (h_{pa}), product inventory cost (h_p), shortage cost (S), and disposal cost (D), to be changeable. Other parameters are assumed to be constant and are set as follows λ_m=2.5; λ_{pa}=2.5; λ_{dis}=1; λ_d=6; M_p=5; and M_{pa}=4. We assign values randomly selected from integers between 1 and 20 to the five changeable parameters. Using this method, 30 different scenarios are randomly obtained and each scenario is labeled by (S,D,h_m,h_p,h_{pa}).

Table 1. Simulated annealing algorithm for the inventory control policy parameters optimization.

1. [Set initial control policy parameters configuration.] Set $\alpha \leftarrow 1$, $\beta \leftarrow 1$, $\gamma \leftarrow 1$, and $Q \leftarrow 1$.
2. [Set initial temperature T_0.] Set $T \leftarrow 0.5$.
3. [Initialize step for number of iterations for terminating algorithm.] Set $S \leftarrow 0$.
4. [Initialize step for number of iterations in each temperature.] Set $I \leftarrow 0$.
5. [Create new configurations of control policy parameters.]
   a) [Create a copy of control policy parameters configuration.] Set $\alpha' \leftarrow \alpha$, $\beta' \leftarrow \beta$, $\gamma' \leftarrow \gamma$, and $Q' \leftarrow Q$.
   b) [Create 2 numbers: increase value and decrease value.] Set increase value $\leftarrow u$, and decrease value $\leftarrow l$.
   c) [Determine a parameter, among four control policy parameters ($\alpha, \beta, \gamma,$ *and* Q), to modify.] Set $Z_v \leftarrow$ [rand(1…4)].
   d) [Determine whether the selected parameter will increase or decrease.] Set $b \leftarrow$ [rand(0,1)].
   e) If $Z_v = 1$, then
      if $b = 1$; set $\alpha \leftarrow \alpha + u$,
      if $b = 0$; set $\alpha \leftarrow \alpha - l$,
      In addition, if $\alpha \leq 0$; set $\alpha \leftarrow 1$.
   f) If $Z_v = 2$, then
      if $b = 1$; set $\beta \leftarrow \beta + u$,
      if b = 0; set $\beta \leftarrow \beta - l$,
      In addition, if $\beta \leq 0$; set $\beta \leftarrow 1$.
   g) If $Z_v = 3$, then
      if $b = 1$; set $\gamma \leftarrow \gamma + u$,
      if b = 0; set $\gamma \leftarrow \gamma - l$,
      In addition, if $\gamma \leq 0$; set $\gamma \leftarrow 1$.
   h) If $Z_v = 4$, then
      if $b = 1$; set $Q \leftarrow Q + u$,
      if b = 0; set $Q \leftarrow Q - l$,
      In addition, if $Q > \alpha$; set $Q \leftarrow \alpha$.
6. [Calculate energy differential.] Set $\Delta E \leftarrow Z(\alpha, \beta, \gamma, Q) - Z(\alpha', \beta', \gamma', Q')$.
7. [Decide upon acceptance of new configuration.] Accept new configuration, if $\Delta E > 0$, or, following the Boltzmann probability distribution, if $\Delta E < 0$ and $exp\left(\frac{-\Delta E}{T}\right) >$ rand(0…1), set $\alpha \leftarrow \alpha'$, $\beta \leftarrow \beta'$, $\gamma \leftarrow \gamma'$, and $Q \leftarrow Q'$.
8. [Repeat for current temperature.] Set $I \leftarrow I + 1$. If $I <$ maximum number of iterations in each temperature, go to step 4.
9. [Lower the annealing temperature.] Set $T \leftarrow c \times T$ ($0 < c < 1$).
10. [Check if progress has been made.] Set $S \leftarrow S + 1$. If $S <$ maximum number of iterations, go to step 4; otherwise, stop.

We express our finding in Table 2 using *%deviation* between two methods' cost function in which *% deviation* $=\frac{|Z_{SA} - Z_{GA}|}{Z_{GA}}$, where the cost function obtained from SA is denoted by Z_{SA} and the cost function obtained from genetic algorithm is denoted by Z_{GA}. We use the following statistical hypothesis test to compare our proposed SA with the real GA.

$$\begin{cases} H_0: \mu_{Z_{SA}} \leq \mu_{Z_{GA}} \\ H_1: \mu_{Z_{SA}} > \mu_{Z_{GA}} \end{cases}$$

The result of proposed statistical hypothesis test indicates a failure to reject the null hypothesis at 95% confidence level. Therefore, it is inferred that the proposed SA works better than GA. Also, based on the above statistical hypothesis test, there is no significant difference between the results of two mentioned methods. Consequently, SA is a convenient, reliable, and effective approach for our problem.

Table 2. The results obtained by SA and GA

Scenario Number	Scenario	SA			GA			% deviation
		Z_{SA}	$(\alpha_{SA},\beta_{SA},\gamma_{SA},Q_{SA})$	time	Z_{GA}	$(\alpha_{GA},\beta_{GA},\gamma_{GA},Q_{GA})$	time	
1	(14,3,7,11,10)	70.7035	(2,2,3,1)	38.08	74.1904	(2,3,3,2)	37.08	4.699934
2	(13,9,5,12,14)	85.7106	(2,2,3,1)	36.36	88.552	(3,2,4,2)	36.94	3.208736
3	(6,19,13,7,5)	93.9671	(2,2,4,1)	37.83	94.0022	(2,2,3,1)	36.75	0.03734
4	(4,17,14,11,7)	90.9575	(2,2,2,1)	37.73	101.6422	(3,2,5,2)	37.28	10.51207
5	(2,2,8,7,4)	29.8276	(1,2,1,1)	40.84	31.4553	(1,2,3,1)	38.02	5.174645
6	(3,14,6,1,7)	61.483	(3,3,4,1)	36.00	64.4805	(6,2,5,2)	34.67	4.648692
7	(3,7,6,11,14)	50.6801	(2,1,2,1)	34.65	53.8952	(3,1,2,2)	35.51	5.965466
8	(16,9,5,5,3)	65.9818	(3,3,7,1)	36.14	67.0597	(4,4,6,1)	36.17	1.607374
9	(4,9,3,3,11)	49.5954	(3,2,6,1)	35.68	51.9829	(3,2,2,1)	36.07	4.592856
10	(18,17,15,14,6)	122.5864	(2,3,4,1)	30.70	132.904	(2,3,5,2)	36.38	7.763197
11	(10,4,6,11,13)	65.9017	(2,2,3,1)	33.64	68.3731	(3,2,2,1)	36.13	3.614579
12	(10,9,2,3,12)	60.0421	(4,2,8,1)	34.70	62.8742	(5,3,6,1)	36.94	4.504391
13	(1,14,13,2,14)	67.0154	(2,1,1,1)	33.86	74.6019	(2,1,5,2)	36.12	10.16931
14	(13,7,8,13,11)	82.8988	(2,2,3,1)	37.41	83.0391	(2,2,2,1)	35.82	0.168957
15	(12,1,4,13,4)	48.1349	(1,3,2,1)	37.12	50.32	(1,4,2,1)	35.81	4.342409
16	(18,2,7,5,11)	68.8674	(1,3,7,1)	37.90	72.6353	(1,3,3,1)	35.33	5.187423
17	(14,2,13,11,14)	78.157	(1,2,3,1)	36.70	86.4889	(2,3,4,1)	36.30	9.633491
18	(10,2,13,2,12)	56.99	(1,2,9,1)	37.09	73.4604	(3,2,3,2)	36.18	22.42079
19	(20,14,6,4,3)	80.5711	(3,4,10,1)	34.88	88.5711	(4,4,4,2)	35.72	9.032292
20	(5,12,4,10,13)	69.5019	(3,2,3,1)	38.30	69.6883	(3,1,4,1)	35.73	0.267477
21	(7,19,14,11,1)	94.3215	(2,4,3,1)	36.90	94.6442	(2,5,3,1)	35.46	0.340961
22	(13,19,13,12,3)	108.2062	(2,3,3,1)	37.32	113.2094	(4,3,4,1)	32.84	4.419421
23	(4,12,11,2,15)	73.311	(2,2,9,1)	37.58	80.0305	(3,2,5,2)	36.39	8.396174
24	(7,4,1,9,5)	39.317	(3,2,2,1)	38.95	41.231	(6,3,3,1)	36.71	4.642138
25	(19,10,2,13,3)	77.1694	(4,3,4,1)	39.28	79.4728	(2,3,3,1)	36.47	2.89835
26	(6,1,9,7,4)	36.657	(1,2,2,1)	37.40	37.4607	(1,3,2,1)	36.56	2.145448
27	(17,20,4,6,7)	95.5097	(4,3,7,1)	38.82	103.4197	(4,2,5,2)	37.08	7.648446
28	(17,4,15,11,3)	78.333	(1,3,3,1)	40.60	83.1584	(1,5,6,1)	36.88	5.802661
29	(7,3,1,12,4)	37.2784	(3,2,1,1)	40.32	38.2096	(5,2,2,2)	36.70	2.437084
30	(15,12,2,10,13)	86.8729	(4,2,4,1)	40.91	90.3022	(4,3,5,2)	39.26	3.797582

6. Sensitivity Analysis

In this section, sensitivity of the model to shortage cost, holding costs, disposal cost and demand rate is analyzed. To do these analyses, we assume the following parameters to be fixed as follows: h_m=2, h_{pa}=4, h_p=5, S=5, D=5, λ_m=2.5, λ_{pa}=2.5, λ_{dis}=1, λ_d=6, M_p=5, and M_{pa}=4. Parameters S and D are two other important parameters to analyze. The results of increasing in these two parameters are depicted in Fig. 2.

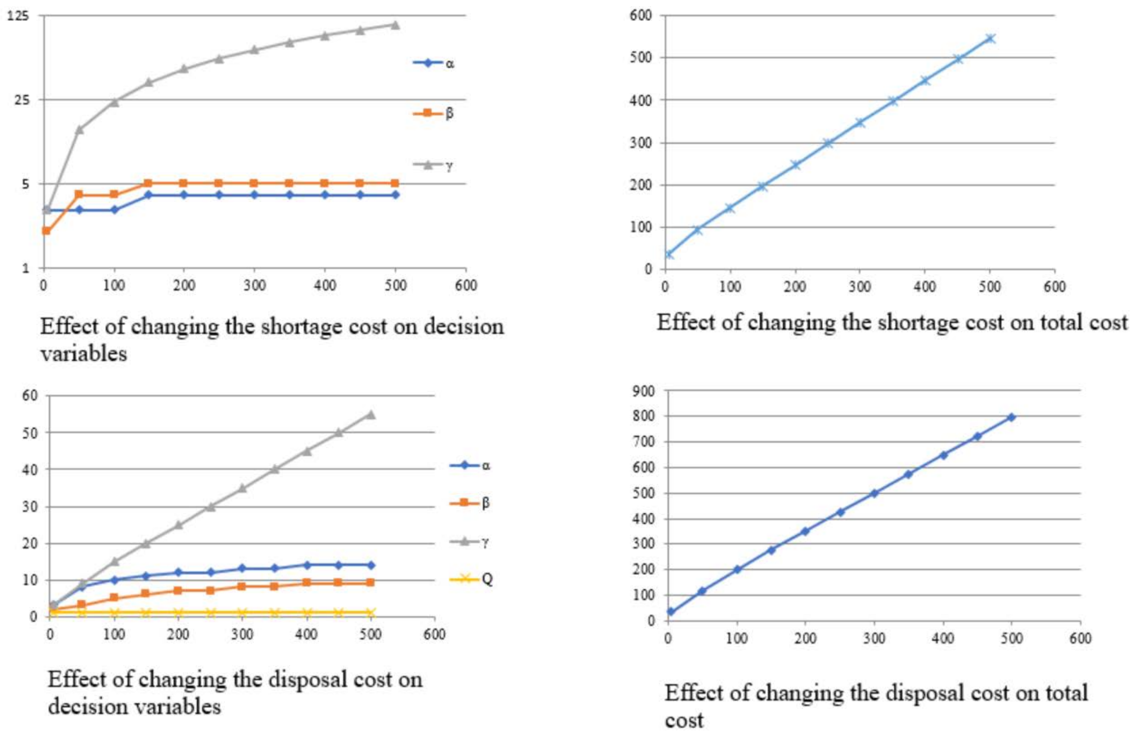

Fig 2. Effect of changing parameters *S* and *D* on decision variables (*α*, *β* and *γ*) and total cost

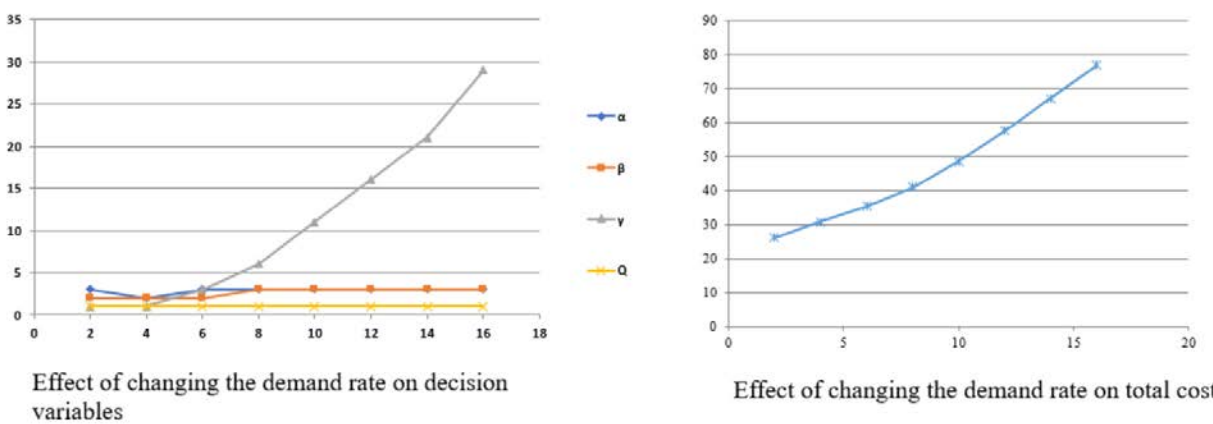

Fig 3. Effect of changing parameters λ_d on decision variables (*α*, *β* and *γ*) and total cost

When *S* increases the remanufacturing system needs to hold more inventory at the product storage, so the optimum value of *γ* increases to overcome the high shortage cost. When disposal cost per item increases the optimum values of *α*, *β*, and *γ* increase. Obviously, when *D* increases the remanufacturing system tends to decrease the number of disposed items. To aim this goal, the system tries to decreases the probability of fullness in the martial storage and part storage by increasing the capacities of these storages. Consequently, the optimum values of all three storages increase. The increase in γ is also decreases the probability of fullness in the martial storage and part storage. It is because when γ increases the portion of time that the part storage is full due to the fullness of product storage decrees.

Fig. 3 depict the effect of changing in λ_d. When demand intensity increases, the remanufacturing system tends to hold more inventory in the product storage, similar to the case that S increases. So, it mostly affects γ. It is worth mentioning that the increase in λ_d has no effect on α, because lead time is assumed to be negligible.

7. Conclusion

In this paper, we investigated a remanufacturing inventory system with stochastic decomposition process. The understudy remanufacturing system is a queuing network system that its close-form steady-state probabilities are unknown in the literature. The main contributions of this paper to the literature are developing the closed-form steady-state probabilities and as a result the closed-form cost function of this remanufacturing system. In stochastic inventory models it is always a challenge to derive the closed-form cost function (Sajadifar and Pourghannad, 2012). Having the closed-form cost function on-hand not only overcomes the round-off errors of numerical examples, but also it facilitates the analysis of cost function and it also increases the efficiency of possible algorithms to optimize the cost function by reducing the run time. To formulate the closed-form cost function of this system, we partitioned the system into four blocks. The cost function of each block is formulated by using conditional probability and queuing network analyses. Finally, the cost function of the system is derived by adding up the cost function of four blocks.

Furthermore, an SA algorithm is developed to generate the near optimum values of inventory control policy. Through the comparison between our proposed SA and an existing real GA developed by Roy et al. (2008), we proved that the answers provided by SA are reliable. Future studies may focus on more advanced solution procedures such as Neural network (Avsar and Aliabadi, 2015), agent-based algorithms (Aliabadi *et al*., 2017). Besides, in section 6 some numerical investigations are presented to provide some managerial insight to the problem. The results indicate that when shortage cost or the demand rate rises, increase of the capacity of product storage is recommended to keep total operation cost as low as possible. Also, to overcome the effect of increase of disposal cost, one needs to increase the capacities of all storages.

For future work in this area, researches can focus on deriving the cost function with considering setup costs for part and product manufacturing processes. Also, considering the non-zero lead time for material purchase process is another interesting issue. Furthermore, the current works usually focus only on inventory models. One interesting direction is integrating the inventory model with finding the optimal location for warehouses (for example by a similar model as in Shahraki *et al*., 2015). Given that return of product induces more transportation, dose the manufacturing really reduce the emissions and lead to sustainability? To do this, one needs to model the supply chain transportation network and its emissions (see Shahraki and Turkay, 2014).

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Appendix

In this appendix, values of $\pi^{p}(\gamma)$, $\pi^{pa}(0)$, $\pi^{pa}(\beta)$, and $1-\pi^{p}(0)$ that are used to formulate the closed-form total cost function of understudy remanufacturing system are calculated. Also, the approach used to derive these steady-state probabilities is explained.

The steady-state probabilities of $\pi^{p}(\gamma)$, $\pi^{pa}(0)$, $\pi^{pa}(\beta)$, and $1-\pi^{p}(0)$ are calculated as follows:

$$\pi^{p}(\gamma) = \frac{A \times C}{1 - A \times B + A \times C} \tag{A.1}$$

$$\pi^{pa}(0) = 1 - \frac{\pi^{p}(\gamma)}{A} \tag{A.2}$$

$$\pi^{pa}(\beta) = \left(1 - \pi^{p}(\gamma)\right) \times E \tag{A.3}$$

$$1 - \pi^{p}(0) = \pi^{pa}(0) \times F + \left(1 - \pi^{pa}(0)\right) \times G \tag{A.4}$$

where $A = \pi^{p}\left(\gamma \middle| \lambda = \frac{1}{M_P}, \mu = \lambda_d\right) = \frac{\left(1-\frac{1}{\lambda_d M_p}\right)\left(\frac{1}{\lambda_d M_p}\right)^{\gamma}}{1-\left(\frac{1}{\lambda_d M_p}\right)^{\gamma+2}}$, $B = 1 - \pi^{pa}\left(0 \middle| \lambda = \lambda_{pa} + \frac{1}{M_{pa}}, \mu = 0\right) = 1$,

$C = 1 - \pi^{pa}\left(0 \middle| \lambda = \lambda_{pa} + \frac{1}{M_{pa}}, \mu = \frac{1}{M_p}\right) = 1 - \frac{\left(1-\frac{\lambda_{pa}+\frac{1}{M_{Pa}}}{\frac{1}{M_P}}\right)}{1-\left(\frac{\lambda_{pa}+\frac{1}{M_{Pa}}}{\frac{1}{M_P}}\right)^{\beta+2}}$, $D = \pi^{pa}\left(\beta \middle| \lambda = \lambda_{pa} + \frac{1}{M_{pa}}, \mu = 0\right) = 0$,

$E = \pi^{pa}\left(\beta \middle| \lambda = \lambda_{pa} + \frac{1}{M_{pa}}, \mu = \frac{1}{M_p}\right) = \frac{\left(1-\frac{\lambda_{pa}+\frac{1}{M_{Pa}}}{\frac{1}{M_P}}\right)\left(\frac{\lambda_{pa}+\frac{1}{M_{Pa}}}{\frac{1}{M_P}}\right)^{\beta}}{1-\left(\frac{\lambda_{pa}+\frac{1}{M_{Pa}}}{\frac{1}{M_P}}\right)^{\beta+2}}$, $F = 1 - \pi^{p}(0 | \lambda = 0, \mu = \lambda_d) = 1$,

$G = 1 - \pi^{p}\left(0 \middle| \lambda = \frac{1}{M_P}, \mu = \lambda_d\right) = 1 - \frac{\left(1-\frac{1}{\lambda_d M_p}\right)}{1-\left(\frac{1}{\lambda_d M_p}\right)^{\gamma+2}}$.

*Proof***:** In the understudy remanufacturing system, as one can infer form Fig 1, input and output rates of each block depends on the fullness or emptiness of other blocks. Considering this fact, steady-state probabilities can be formulated by conditioning on the fullness or emptiness of the beforehand or afterward storage. In this appendix, by using conditional probability, we compute $\pi^{p}(\gamma)$, $\pi^{pa}(0)$, $\pi^{pa}(\beta)$, and $1-\pi^{p}(0)$ as shown by Eqs. (A.5) to (A.8). Other steady-state probabilities can be obtained directly using classical queuing theory, *e.g.* $\pi^{p}(\gamma | part\ storage\ is\ not\ empty)$ is calculated from the existing closed-form relations for steady-state probabilities of a clasic *M*/*M*/1/ γ queuing system.

$$\pi^{p}(\gamma) = \left(1 - \pi^{pa}(0)\right) \times \pi^{p}\left(\gamma \middle| \begin{matrix}\text{part storage}\\ \text{is not empty}\end{matrix}\right) + \pi^{pa}(0) \times \pi^{p}\left(\gamma \middle| \begin{matrix}\text{part storage}\\ \text{is empty}\end{matrix}\right) \tag{A.5}$$

$$1 - \pi^{pa}(0) = \pi^{p}(\gamma) \times \pi^{pa}\left(\begin{matrix}\text{part storage}\\ \text{is not empty}\end{matrix} \middle| \begin{matrix}\text{product storage}\\ \text{is full}\end{matrix}\right) + \left(1 - \pi^{p}(\gamma)\right) \times \pi^{pa}\left(\begin{matrix}\text{part storage}\\ \text{is not empty}\end{matrix} \middle| \begin{matrix}\text{product storage}\\ \text{is not full}\end{matrix}\right) \tag{A.6}$$

$$\pi^{pa}(\beta) = \pi^{p}(\gamma) \times \pi^{pa}\left(\beta \middle| \begin{matrix}\text{product storage}\\ \text{is full}\end{matrix}\right) + \left(1 - \pi^{p}(\gamma)\right) \times \pi^{pa}\left(\beta \middle| \begin{matrix}\text{product storage}\\ \text{is not full}\end{matrix}\right) \tag{A.7}$$

$$1 - \pi^{p}(0) = \pi^{pa}(0) \times \pi^{p}\left(\begin{matrix}\text{product storage}\\ \text{is not empty}\end{matrix} \middle| \begin{matrix}\text{part storage}\\ \text{is empty}\end{matrix}\right) + \left(1 - \pi^{pa}(0)\right) \times \pi^{p}\left(\begin{matrix}\text{product storage}\\ \text{is not empty}\end{matrix} \middle| \begin{matrix}\text{part storage}\\ \text{is not empty}\end{matrix}\right). \tag{A.8}$$

where,

$\pi^{p}\left(\gamma \middle| \begin{matrix}\text{part storage}\\ \text{is not empty}\end{matrix}\right) \rightarrow \pi^{p}\left(\gamma \middle| \lambda = \frac{1}{M_P}, \mu = \lambda_d\right)$; Let's call this value ***A***.

$\pi^{p}\left(\gamma \middle| \begin{matrix}\text{part storage}\\ \text{is empty}\end{matrix}\right) \rightarrow \pi^{p}(\gamma | \lambda = 0, \mu = \lambda_d)$; This value is equal to 1.

$\pi^{pa}\left(\begin{matrix}\text{part storage}\\ \text{is not empty}\end{matrix}\middle|\begin{matrix}\text{product storage}\\ \text{is full}\end{matrix}\right) \rightarrow 1-\pi^{pa}\left(0\middle|\lambda=\lambda_{pa}+\frac{1}{M_{pa}}, \mu=0\right)$; Let's call this value ***B***.

$\pi^{pa}\left(\begin{matrix}\text{part storage}\\ \text{is not empty}\end{matrix}\middle|\begin{matrix}\text{product storage}\\ \text{is not full}\end{matrix}\right) \rightarrow 1-\pi^{pa}\left(0\middle|\lambda=\lambda_{pa}+\frac{1}{M_{pa}}, \mu=\frac{1}{M_p}\right)$; Let's call this value ***C***.

$\pi^{pa}\left(\beta\middle|\begin{matrix}\text{product storage}\\ \text{is full}\end{matrix}\right) \rightarrow \pi^{pa}\left(\beta\middle|\lambda=\lambda_{pa}+\frac{1}{M_{pa}}, \mu=0\right)$; Let's call this value ***D***.

$\pi^{pa}\left(\beta\middle|\begin{matrix}\text{product storage}\\ \text{is not full}\end{matrix}\right) \rightarrow \pi^{pa}\left(\beta\middle|\lambda=\lambda_{pa}+\frac{1}{M_{pa}}, \mu=\frac{1}{M_p}\right)$; Let's call this value ***E***.

$\pi^{p}\left(\begin{matrix}\text{product storage}\\ \text{is not empty}\end{matrix}\middle|\begin{matrix}\text{part storage}\\ \text{is empty}\end{matrix}\right) \rightarrow 1-\pi^{p}(0|\lambda=0, \mu=\lambda_d)$; Let's call this value ***F***.

$\pi^{p}\left(\begin{matrix}\text{product storage}\\ \text{is not empty}\end{matrix}\middle|\begin{matrix}\text{part storage}\\ \text{is not empty}\end{matrix}\right) \rightarrow 1-\pi^{p}\left(0\middle|\lambda=\frac{1}{M_P}, \mu=\lambda_d\right)$; Let's call this value ***G***.

By solving simultaneous equations (A.5) to (A.8), we get the steady-state probabilities as in equations (A.1) to (A.4).

Biography:

Narges Shahraki has completed her Ph.D. program in Industrial Engineering and Operations Management at Koc University in August 2015. Currently, she is a Master Scheduler at Michael foods Inc. Her current research focuses on production and inventory analysis.