Game of Patterns: an approach for solving mixed integer nonlinear optimization problems

Ebert Brea
Department of Electronic, Computing and Control,
School of Electrical Engineering, Faculty of Engineering, Central University of Venezuela,
Caracas 1053, Venezuela
ebert.brea@ucv.ve, ebertbrea@gmail.com

Abstract

This article presents a novel direct search method of optimization, Game of Patterns (GoP) method, so-called by the author, for solving unconstrained mixed integer nonlinear optimization problems. The GoP method is based on: a set of η random patterns search, which initially forms the set of η active players, namely, $\left\{\mathcal{H}_1^{[0]}, \ldots \mathcal{H}_\eta^0\right\}$; and a game rule framework. At each kth game round, each eth active player $\mathcal{H}_e^{[k]}$ will explore inside his own pattern at the current kth round. The strategy of each eth player is given by a random quantity $S_e^{[k]}$, which will allow each player to explore inside his own pattern by a set of $S_e^{[k]}$trial points. At the beginning of each kth game round, each active eth player bets $S_e^{[k]}$ according to his budge M_e for each round. At the end of each kth round, the player that has identified the best objective function value is considered the winner of the kth round, therefore the rest of active players must pay off to the winner their bets. This process is recurrently repeated until that has been disqualified $\eta - 1$ players from the game. It is worthwhile to point out that any player will be disqualified if his balance account $b_e^{[k]}$ is less than his budget M_e for the current kth round.

Keywords
Mixed integer nonlinear optimization, direct search optimization, randomized pattern search algorithm, game theory, frees derivative optimization method.

1. Introduction
Nowadays, optimization problems in business, medicine, engineering sciences, and applied sciences nowadays are more and more complex, which can involve both continuous and integer variables are able to describe many real world problems.

Furthermore, the problems of optimum designs are often studied in engineering science. Examples of this kind of problems are presented by Brea (2009, 2013), who describes a set of several engineering problems. Floudas (1995) also presents a wide set of optimum designs for chemistry process plants under a viewpoint of optimization. Moreover, Yang and coworker (2012) present the approach of game theory for using in multi-objective optimization problems, and Audet, et al., (2001) have proposed pattern search method for locally solving constrained mixed integer nonlinear problems.

In our case, we shall use both an approach of game theory and a random pattern search algorithm, the Mixed Integer Randomized Pattern Search Algorithm (MIRPA) (Brea, 2015), for proposing the design of a novel direct search optimization method, Game of Patterns (GoP) method, so-called by the author, allows us to identify at less, a local optimum solution to unconstrained mixed integer nonlinear problems. Much has been written on optimization and the theory of game. Nevertheless, our focus has been on using the theory of game for including a theoretical framework, what will be defined as a new heuristics for identifying local solutions to unconstrained mixed integer nonlinear optimization problems. In this case, a zero-sum game framework was defined, therefore, each player

chooses his own strategy regardless of the opponent players, and at each round game, when one player wins, the rest of players must pay off their bets to the winner.

This new approach takes into account the kth set $\mathcal{A}^{[k]}$ of $\eta_A^{[k]}$ (≥2) active players, defined at each kth round by the active patterns of search $\left\{\mathcal{H}_e^{[k]}\right\}_{e\in A^{[k]}}$, which are under a rule of game. Each eth player is represented by a randomized pattern of search, which allows it to explore through a set of $S_e^{[k]}$ trial points in the eth randomized pattern of search. The strategy of each eth player at each kth game round is given by a random number $S_e^{[k]}$ of trial points in the randomized pattern of search, wherein can be evaluated the objective function of the problem. This random number of trial points depends on the budget M_e of each eth player at the kth game round and the available balance account $b_e^{[k]}$ at the kth round.

At the beginning of each kth round, each active eth player makes a bet $S_e^{[k]}$ according to his budget M_e for each round and balance account $b_e^{[k]}$. The player that has identified the best objective function value is considered the winner of the kth round, therefore the rest of active players must pay off their bets to the winner. We have assumed that each player e randomly choose his bet according to a uniform distribution between $(n+m)$ and $2(n+m)$, However, this strategy scheme can be changed by another function distribution.

This process is recurrently repeated until $\mathcal{A}^{[k]}$ has become a singleton set at any kth round. It is worthwhile to point out that any player is disqualified at any round, if his balance account $b_e^{[k]}$ is less than his budget M_e. On the convergence of each player has previously been presented by Brea (2015), in his study of the Mixed Integer Randomized Pattern Search Algorithm (MIRPSA) for proving the convergence of the MIRPSA to a local minimum.

The remainder of this article is organized as follows. In the next section 2, we briefly describe our unconstrained mixed integer nonlinear problems, which an explicit mathematical expression of the objective function is not necessarily available, what could require the evaluation of the objective function by simulation. In section 3, we give a short theoretical background of a generic framework of game. On the game rule and a mathematical formulation of our game of patterns is shown in section 4. In section 5, we offer an explanation of the GoP method, and we present the pseudo code of the method. In section 6 are presented a set of numerical examples in order to measure the performance of the GoP to some problems. Finally, in section 7, we give some concluding remarks and we shall propose future research.

2. The problem

Consider the following unconstrained mixed integer nonlinear problem:

$$\min_{(x,y)\in\mathbb{R}^n\times\mathbb{Z}^m} f(x,y), \tag{1}$$

where $f(x,y)\colon \mathbb{R}^n\times\mathbb{Z}^m \to \mathbb{R}$ is a mixed integer nonlinear objective function, which is not necessarily available an explicit mathematical expression, and $x\in\mathbb{R}^n$ and $y\in\mathbb{Z}^m$ are the decision variables of our problem.

Nevertheless, constrained mixed integer nonlinear problems often arise out from practical applications, which are frequently studied in the engineering design field, wherein there exist both real and integer constrained variables. Hence, penalty function method approach is considered for solving such problems, when direct search optimization methods for solving unconstrained problems are used for solving constrained optimization problems. A practical example of penalty function method is presented by Brea (2013).

3. A Short Theoretical Background

Let Γ be a game in general terms defined by the structure $\{\mathcal{H}_e;\ S_e;\ Q_e\}_{e=1}^{\eta}$, where $\mathcal{H}_e$ is each eth player, S_e is the strategy of each eth player, and Q_e is the pay off function of each eth player, for all $e\in\{1,2,\dots\eta\}$. The pay off function is given by $Q_e\colon\times_{e=1}^{\eta} S_e \to \mathbb{R}$ (Mazalov, 2014).

In our case, the players $\{\mathcal{H}_e\}_{e=1}^{\eta}$ are mixed integer randomized pattern searches defined in (Brea, 2015), which each one of them accounts with two basic operations for exploring the Euclidean field $\mathbb{R}^n\times\mathbb{Z}^m$, and so identifies a local minimum at least. This fact has been studied by Brea (2016), who has reported his study on the performance of the

MIRPSA. An explanation of these operations and a detailed analysis of the MIRPSA is presented by Brea (2015). Nevertheless, the idea of using a set of mixed integer randomized pattern searches in a framework of game allows us to identify at least a local minimum of our problem is introduced in this article. However, even though it has not been theoretically proved, the GoP could be used to identify the global minimum of mixed integer problems, what has been verified through some numerical examples.

The payment of each *e*th player is defined as a function, which depends on the number of objective function evaluation, what will be the strategy $S_e^{[k]}$of each *e*th player at each *k*th game round. In our case, a game will be a countable set of rounds, which will finish when only one of the players $\mathcal{H}_e^{[\cdot]}$ survives, because each player can be eliminated due to that lose all his account points.

4. On the Game of Pattern Searches

Assume η initial active randomized pattern searches $\mathcal{H}_e^{[0]}$ for all e=1,…,η, which represents all players of the game Γ and each one respectively has an initial positive balance account $b_e^{[0]}$. The bet of each *e*th player at each *k*th round is $S_e^{[k]}$, what here represents the strategies of the active player $e \in \mathcal{A}^{[k]}$at each round *k*. Also, let $\mathcal{H}_e^{[k]}$denote for each $e \in \overline{\mathcal{A}^{[k]}}$ the *e*th disqualified player at each *k*th round.

According to the game rule, any *e*th active player $\mathcal{H}_e^{[k]}$ will become disqualified player $\overline{\mathcal{H}_e^{[k]}}$ at any *k*th round, if his account balance $b_e^{[k+1]}$, after the *k*th round, becomes less than the level of bet $M_e = n + m$.

We then define the strategy of each *e*th player as

$$S_e^{[k]} = \begin{cases} \bar{u}(M_e, 2M_e), & \forall e \in \mathcal{A}^{[k]}; \\ 0, & \forall e \in \overline{\mathcal{A}^{[k]}} \end{cases} \tag{2}$$

where, $\bar{u}(.,.)$ a discrete uniform distribution, what allows the method to determinate the quantity of trial points that belongs to the *e*th active random pattern of search $\mathcal{H}_e^{[k]}$; or the strategy of each disqualified *e*th player is naught. However, this strategy is enough flexible, because this random integer number can be defined by another discrete distribution. We must point out that each *e*th player is active at each *k*th round, if his current balance $b_e^{[k]}$ is more or equal than his level of bet M_e this is, for each $e \in \mathcal{A}^{[k]}$, $b_e^{[k]} \geq M_e$.
Let c_e denote the unit cost of evaluating the objective function for each *e*th player, which will here be constant at each *k*th round. In our case, the unit cost will be considered equal to 1 per function evaluation for each *e*th player. Hence, the pay off $Q_e^{[k]}$ of each *e*th player at each *k*th round takes the form

$$Q_e^{[k]} = S_e^{[k]}, \forall\, 1 \leq e \leq \eta \tag{3}$$

Then, a game structure for all *k*th game round is defined by $\Gamma^{[k]} = \{\mathcal{H}_e^{[k]}; S_e^{[k]}; Q_e^{[k]}\}_{e \in \mathcal{A}_e^{[k]}}$, what allows us to define the game structure $\Gamma = \{\Gamma^{\{k\}}\}_{k=0}^{k=\hat{k}}$, where $\hat{k}$ is the last game round.

We say that the winner of each *k*th round, denoted as $\hat{e}^{[k]}$ is the random choice *e*th active player from the set of active players has tied in the identification of the best objective function value, at the *k*th round, this is,

$$\hat{e}^{[k]} = rdm\left(\arg\min_{e \in \mathcal{A}_e^{[k]}} \hat{f}_e^{[k]}(x, y)\right), \tag{4}$$

where $\hat{f}_e^{[k]}(x, y)$ is the best objective function value yielded by the *e*th players $\mathcal{H}_e^{[k]}$ at the *k*th round.

It is worthwhile pointing out that each *e*th active player has his own account balance $b_e^{[k]}$, which is updated at each *k*th round, and is given by

$$b_e^{[k+1]} = \begin{cases} b_e^{[k]} + \sum_{e \in \mathcal{A}^{[k]} \setminus \hat{e}^{[k]}} S_e^{[k]}, & \forall e = \hat{e}, e \in \mathcal{A}^{[k]}; \\ b_e^{[k]} - S_e^{[k]}, & \forall e \neq \hat{e}, e \in \mathcal{A}^{[k]}, \end{cases} \tag{5}$$

where $b_e^{[0]}$ is the initial budget of each *e*th player, and $\hat{e}^{[k]}$ given by (4) at each *k*th round.

At each kth round and each eth player targets to improve a same objective function $f(x,y)$, with an associate cost given by the bet of each player at each kth round.

5. The Game of Patterns Method

Here we shall show the GoP method, displayed in Figs. 1 and 2 , which are based on a set of players given by a set of active random patterns $\{\mathcal{H}_e^{[k]}\}_{e=1}^{\eta}$ for all $e \in \mathcal{A}^{[k]}$ at each kth game round.

Preamble of the Game of Patterns

Initialization

Let $f(x,y) : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{R}$ be given.

Let $v_{0,e} = (v_{0,e}^{(1)}, \ldots, v_{0,e}^{(n)}, v_{0,e}^{(n+1)}, \ldots, v_{0,e}^{(n+m)})^t = (x_e^{(1)}, \ldots, x_e^{(n)}, y_e^{(1)}, \ldots, y_e^{(m)})^t \in \mathbb{R}^n \times \mathbb{Z}^m$ be each eth initial pattern search center guess of each eth pattern search.

Let k be the round counter number.

Let $M_e = n + m$ be a minimum number of trial points belonging to each eth pattern search at each round, which allows us to define the strategy of each eth player $\mathcal{H}_e^{[k]}$.

Let η be the number of players, which must be more than one.

Let $\delta_e^{[0]} \in \mathbb{R}_+$ be the initial parameter of continuous uniform distribution.

Let $\Delta_e^{[0]} \in \mathbb{N}_+$ be the initial parameter of discrete uniform distribution.

Let $n_A^{[0]} = \eta$ be an initial number of active pattern search at initial round, where $\eta \geq 2$.

Let $b_e^{[0]} > M_e$ be the initial account balance of each eth player $\mathcal{H}_e^{[0]}$.

Let $\mathcal{A}^{[0]} = \{1, 2, \ldots, \eta\}$ be the initial set of active players $\mathcal{H}_e^{[0]}$ for all $e \in \{1, 2, \ldots, \eta\}$.

Declare

$z_e = (z_e^{(1)}, \ldots, z_e^{(n)}, z_e^{(n+1)}, \ldots, z_e^{(n+m)})^t \in \mathbb{R}^n \times \mathbb{Z}^m$ as any trial point of each eth pattern search.

$c_e^{[k]} = (c_e^{[k]^{(1)}}, \ldots, c_e^{[k]^{(n)}}, c_e^{[k]^{(n+1)}}, \ldots, c_e^{[k]^{(n+m)}})^t \in \mathbb{R}^n \times \mathbb{Z}^m$ as temporary point of each eth pattern search.

Data:

Let $\alpha \in \mathbb{R} | 0 < \alpha < 1$ be the real shrinking parameter of each eth pattern search;

Let $\beta \in \mathbb{R} | 0 < \beta < 1$ be the integer shrinking parameter of each eth pattern search;

Let $\varepsilon \in \mathbb{R}_+$ be the tolerance used to test stopping rule of any pattern search.

Figure 1. Preamble of the Game of Patterns method.

In Fig. 1 we show the preamble of the GoP method, which defines parameters, variables and initial value of the GoP method. It is worthwhile pointing out that in our case, each component of each initial eth pattern search center $c_e^{[k]}$ was randomly fixed according to:

$$c_e^{[k]^{(i)}} = \begin{cases} x_e^{(i)} + u(-\varphi, \varphi), & \forall i = 1, \ldots, n; \\ y_e^{(i)} + \bar{u}(-\Phi, \Phi), & \forall i = n+1, \ldots, m, \end{cases} \tag{6}$$

where $u(\,,)$ is a real uniform distribution, and $\bar{u}(\,,)$ is a discrete uniform distribution, and $\varphi \in \mathbb{R}(> 0)$ and $\Phi \in \mathbb{N}(\geq 1)$ are chosen by the user. On the other hand, Fig. 2 depicts the pseudocode of the iterative block GoP method, which will be presented in the remainder of this section.

Pseudocode of the Game of Patterns loop block

```
k ← 0;
while n_A^[k] ≥ 2 do
    for e ← 1 to η do
        d_e^[k] ← δ_e^[k] + Δ_e^[k];
        if e ∈ 𝒜^[k] then
            S_e^[k] ← ū(M_e, 2M_e)
            Call Procedure EXPLORER(e, k, α, β, c_e^[k], δ_e^[k], Δ_e^[k], S_e^[k])
        else
            S_e^[k] ← 0
        end
    end
    Update ê^[k] = rdm( arg min_{e∈𝒜^[k]} f̂_e^[k](x, y) )
    for e ← 1 to η do
        if e ∈ 𝒜^[k] then
            if e = ê^[k] then  b_e^[k+1] ← b_e^[k] + Σ_{i∈𝒜^[k]\ê^[k]} S_e^[k]  else  b_e^[k+1] ← b_e^[k] − S_e^[k]
            if b_e^[k+1] < M_e then
                Disqualify ℋ_e^[k]
                n_A^[k+1] ← n_A^[k] − 1
                Update 𝒜^[k]
            end
        end
    end
    n_A^[k+1] ← n_A^[k]
    𝒜^[k+1] ← 𝒜^[k]
    k ← k + 1
end
δ_ê^[k] ← δ_ê^[0];
Δ_ê^[k] ← Δ_ê^[0];
d_ê^[k] ← δ_ê^[k] + Δ_ê^[k];
while d_ê^[k] > ε do
    S_e^[k] ← ū(M_e, 2M_e)
    Call Procedure EXPLORER(ê, k, α, β, c_ê^[k], δ_ê^[k], Δ_ê^[k], S_ê^[k])
    k ← k + 1
    d_ê^[k] ← δ_ê^[k] + Δ_ê^[k]
end
Report: the optimum c_{ê^[k−1]}^[k−1], and then terminate.
```

Figure 2. Pseudocode of the Game of Patterns loop block

At each kth iteration (game round) of the GoP method and for all active eth player, the GoP method then calculates how many trial points $S_e^{[k]}$ using (2), and performs the procedure $EXPLORER(e, k, \alpha, \beta, c_e^{[k]}, \delta_e^{[k]}, \Delta_e^{[k]}, S_e^{[k]})$, which is shown in Fig. 3.

Pseudocode of the Procedure EXPLORER $\left(e, k, \alpha, \beta, c_e^{[k]}, \delta_e^{[k]}, \Delta_e^{[k]}, S_e^{[k]}\right)$

Initialization:
Let $s = 0$ be an initial boolean moving indicator.
Declare:
$z = (z^{(1)}, \ldots, z^{(n)}, z^{(n+1)}, \ldots, z^{(n+m)})^t \in \mathbb{R}^n \times \mathbb{Z}^m$ as any trial point.
Given
- e the eth player identification;
- k the the round counter number;
- $\alpha \in \mathbb{R} | 0 < \alpha < 1$ the real shrinking parameter;
- $\beta \in \mathbb{R} | 0 < \beta < 1$ the integer shrinking parameter;
- $c_e^{[k]}$ the center of the eth pattern search (player)at the current kth round;
- $\delta_e^{[k]} \in \mathbb{R}_+$ the parameter of continuous uniform distribution of the eth player at the kth round;
- $\Delta_e^{[k]} \in \mathbb{N}_+$ the parameter of discrete uniform distribution of the eth player at the kth round;
- $S_e^{[k]}$ the strategy of the eth player, which means the number of trial points.

```
for j ← 1 to S_e^[k] do
    for i ← 1 to n do
        z^(i) ← c_e^[k](i) + u(−δ_e^[k], δ_e^[k])
    end
    for i ← n+1 to n+m do
        z^(i) ← c_e^[k](i) + ū(−max(1, Δ_e^[k]), max(1, Δ_e^[k]))
    end
    if f(z) < f(c_e^[k]) then
        s ← 1;
        c_e^[k] ← z;
    end
end
if s = 1 then
    δ_e^[k+1] ← δ_e^[k];
    Δ_e^[k+1] ← Δ_e^[k];
else
    δ_e^[k+1] ← αδ_e^[k];
    Δ_e^[k+1] ← βΔ_e^[k];
end
c_e^[k+1] ← c_e^[k]
Return: c_e^[k+1], δ_e^[k+1], Δ_e^[k+1].
```

Figure 3. Procedure EXPLORER

Otherwise, the GoP method does not allow pattern of search to perform the procedure EXPLORER(), and obviously, for these cases the strategies $S_e^{[k]}$are fixed at 0. It is worth noting that procedure EXPLORER() is just one iteration of the MIRPSA (Brea, 2015).
After this last stage, the GoP method identifies the winner player of the *k*th round $\hat{e}^{[k]}$ among the active players at the *k*th current game round by recognizing of the best objective function using (4), therefore the rest of active players must pay off to the winner their bets.

After pay off stage, the GoP method updates: the set of active players $\mathcal{A}^{[k]}$; the number of active player $n_{\mathcal{A}}^{[k]}$; the index $\hat{e}^{[k]}$ of the winner of the current active set $\mathcal{A}^{[k]}$ at the *k*th current game round; and the round counter number *k*, in order to perform a new iteration, if there exits more than one active random pattern players $\mathcal{H}_e^{[k]}$. The GoP method is iteratively carried out until the set of active random pattern players becomes a singleton set. After this stage, the GoP method runs an additional iterative while process by calling the procedure EXPLORER(), until the pattern measure $d_{\hat{e}}^{[k]}$ of the winner player $\mathcal{H}_{\hat{e}}^{[k]}$ is less than the stopping test tolerance ϵ.
Finally, the GoP method reports $c_{\hat{e}}^{[k-1]}$ as the local solution of our problem.

6. Numerical Examples

A set of numerical examples are here reported. The notation of each minimization problem is shown in the appendix of this article. The tables denote the number of function evaluation (NE), the distance to true point (DTP) and the reported solution of each minimization problem.
The parameters of the GoP methods were fixed at: $\alpha = \beta = 0.9$, $\delta = \Delta = 5$, $\epsilon = 10^{-6}$, $\varphi = \Phi = 10$, the number of sampling equal to 1000, number of players η=5, and the same start point, which was fixed at 10 for all components of $\nu_{0,e}$.

6.1 Goldstein-Price problem

Table 1 shows a statistical summary of the performance of the GoP for 1000 replications for Goldstein-Price problem. From the table we note the GoP method identifies the global minimum for all replications.

Table 1. Statistical summary report of Goldstein-Price problem

	Optimum value	NE	DTP
Average	3.00000000000015	13569.7	9.20E-09
Minimum	2.99999999999992	6506	8.87E-11
Maximum	3.00000000002377	25288	3.03E-07
Range	2.3848922837E-11	18782	3.02E-07

6.2 Audet & Dennis problem

Table 2 displays a statistical summary of the performance of the GoP for 1000 replications for Audet & Dennis problem. Here, the GoP method identifies the global minimum for all replications.

Table 2. Statistical summary report of Audet & Dennis problem

	Optimum value	NE	DTP
Average	-13.9951707839905	8387,5	8.05E-04
Minimum	-14.0000000000000	5653	0.00E+0
Maximum	-13.1780431847197	14617	1.37E-01
Range	0.8219568152803	8964	1.367E-01

6.3 Modified Griewank problem

Table 3 reports a statistical summary of the performance of the GoP for 1000 replications for Modified Griewank problem of dimension $\mathbb{R}^n \times \mathbb{Z}^m$. As we can see, the GoP method identifies the global minimum in most cases.

Table 3. Statistical summary report of Modified Griewank problem

		Opt. value	NE	DTP
n=2, m=2	Average	0.10106215	7796.547	0.5146393
	Minimum	0	5884	5.9723E-10
	Maximum	0.74549191	16525	3.3209256
	Range	0.74549191	10641	3.3209255
n=5, m=5	Average	0.62162712	10972.955	1.7146742
	Minimum	0.00256293	7258	0.05527682
	Maximum	2.00422415	19190	4.8112703
	Range	2.00166122	11932	4.75599348
n=5, m=10	Average	0.87451727	13411.2	1.41896373
	Minimum	0.01509559	8266	0.13430363
	Maximum	2.26325818	26940	5.87207788
	Range	2.24816259	18674	5.73777425
n=10, m=5	Average	1.44654033	12967.381	2.90170741
	Minimum	0.05302396	8221	0.25380868
	Maximum	2.64750672	21311	5.11997786
	Range	2.59448275	13090	4.86616919
n=10, m=10	Average	1.25370734	14769.069	1.81073155
	Minimum	0.05478916	9321	0.25808091
	Maximum	2.87108489	28959	6.01925066
	Range	2.81629573	19638	5.76116975

6.4 W problem

For this problem of n=m=2, the parameters of the GoP methods were fixed at: $\alpha = \beta = 0.9$, $\delta = \Delta = 5$, $\epsilon = 10^{-6}$, $\varphi = \Phi = 100$, the number of sampling equal to 1000, number of players η=5, and the same start point, which was fixed at 0 for all components of $v_{0,e}$. From the table, we can say the GoP method identifies the global minimum.

Table 4. Statistical summary report of W problem

	Optimum value	NE	DTP
Average	-185,215991017	8669,35	13,6867026
Minimum	-186,000000000	5885	1,1435E-09
Maximum	-184,000000000	15956	32
Range	2,000000000	10071	32

6.5 Mixed integer Tang problem

In this case, n=m=2, the parameters of the GoP methods were fixed at: $\alpha = \beta = 0.9$, $\delta = \Delta = 5$, $\epsilon = 10^{-6}$, $\varphi = \Phi = 100$, the number of sampling equal to 1000, number of players η=5, and the same start point, which was fixed at 0 for all components of $v_{0,e}$. From the table, we can say the GoP method identifies the global minimum.

Table 5. Statistical summary report of mixed integer Tang problem

	Optimum value	NE	DTP
Average	-4,61027593	8438,069	0,62757089
Minimum	-4,73094883	5865	0,00542486
Maximum	-3,73830331	16810	5,08207131
Range	0,99264552	10945	5,07664645

From the report, 876 replications yielded a DTP less than 0.013, what allow us to indicate the GoP method satisfactorily identifies in this case the global minimum.

7. Conclusions

The GoP method has shown us to be a good enough heuristic method for identifying at least a local optimal solutions to mixed integer nonlinear optimization problems. Preliminary results have allowed us to figure out the GoP method can be used for solving mixed integer nonlinear global optimization problems. However, we need to study some converge properties of the GoP method first. An experimental scheme will be proposed for tuning the parameters of the GoP method for improving its performance.
In order to keep developing the method, we shall also research others discrete distribution for choosing the strategy of each player for improving the competition among the players.

A. Appendix

A set of mixed integer optimization problems are here shown for testing the GoP. A notation is defined for each problem, which allows us to define the dimension of the problems in $\mathbb{R}^n \times \mathbb{Z}^m$.

A.1 Goldstein-Price problem

Although the Goldstein-Price function is defined in the two-dimension real numeric field, we have used it for testing the GoP method, due to the fact this function has been frequently used to test global optimization algorithms (Shi and Ólafsson, 2000).

Let $g(x)$ denote

$$\begin{aligned} g(x) \quad &= [1 + \left(x^{(1)} + x^{(2)} + 1\right)^2 (19 - 14x^{(1)} + 3{x^{(1)}}^2)^2 - 14x^{(2)} + 6x^{(1)}x^{(2)} + 3{x^{(2)}}^2] \\ &\cdot [30 + \left(2x^{(1)} - 3x^{(2)}\right)^2 \left(18 - 32x^{(1)} + 12{x^{(1)}}^2\right)^2 + 48x^{(2)} - 36x^{(1)}x^{(2)} + 27{x^{(2)}}^2]' \end{aligned} \tag{7}$$

which is our objective function of the constrained optimization problem, subject to: $-2.5 \leq x^{(i)} \leq 2, \forall\, i \in \{1,2\}$.

Using penalty functions, the problem can be expressed as follows:

$$\min_{x \in \mathbb{R}^2} f(x) = g(x) + \sum_{i=1}^{2} k\, p_i(x^{(i)}), \tag{8}$$

where k was fixed at 10^6, and

$$p_i\left(x^{(i)}\right) = \max\left(-2.5 - x^{(i)}, 0\right) + \max\left(x^{(i)} - 2{,}0\right), \forall\, i \in \{1,2\} \tag{9}$$

Solution. The global minimum is located at $\hat{x} = (0, -1)^t$, and a value of $f(\hat{x}) = 3$.

A.2 Audet & Dennis problem

The original problem of Audet and Dennis Jr. (2001) is

$$\min_{x \in \mathbb{R}^2, y \in \mathbb{Z}} f(x, y), \tag{10}$$

subject to

$$-2 \leq x^{(1)} \leq 2;\ -2 \leq x^{(2)} \leq 2;\ y^{(1)} \in \{1,2\}, \tag{11}$$

where

$$f(x,y) = g(x)\left(1 - y^{(1)}\right) + h(x)\, y^{(1)}, \forall\, x \in \mathbb{R}^2, y \in \mathbb{Z}\,, \tag{12}$$

and the function $g(x)$ and $h(x)$ for all $x \in \mathbb{R}^2$ are given by

$$\begin{aligned} g(x) &= {x^{(1)}}^2 + {x^{(2)}}^2; \\ h(x) &= {x^{(1)}}^2 x^{(2)} + x^{(1)}\left(1 - x^{(2)}\right). \end{aligned} \tag{13}$$

Using penalty function, the problem can be expressed as follows:

$$\min_{x\in\mathbb{R}^2} g(x)\left(1 - y^{(1)}\right) + h(x) + \sum_{i=1}^{2} k\, p_i\left(x^{(i)}\right) + k\, p_3(\, y^{(1)}), \tag{14}$$

where k was fixed at 10^3, and

$$\begin{aligned} p_i\left(x^{(i)}\right) &= \max\left(-2 - x^{(i)}, 0\right) + \max\left(x^{(i)} - 2{,}0\right), \forall\, i \in \{1{,}2\}; \\ p_3\left(y^{(1)}\right) &= \max\left(1 - y^{(1)}, 0\right) + \max\left(y^{(1)} - 2{,}0\right). \end{aligned} \tag{15}$$

Solution. The global minimum is located at $\hat{x} = (-2,-2)^t$, and $\hat{y} = 0$, and a value of $f(\hat{x},\hat{y}) = -14$.

A.3 Modified Griewank problem

We have formulated a modified version in the mixed integer numerical field of the Griewank function, which is shown by Fu and coworker (Fu, et al, 2006).

$$\min_{x\in\mathbb{R}^n, y\in\mathbb{Z}^m} f(x,y), \tag{16}$$

where

$$f(x,y) = 2 + \frac{1}{20}\sum_{i=1}^{n} {x^{(i)}}^2 + \frac{1}{20}\sum_{i=1}^{m} {y^{(i)}}^2 - \prod_{i=1}^{n}\cos\left(\frac{2\pi}{5}x^{(i)}\right) - \prod_{i=1}^{n}\cos\left(\frac{2\pi}{5}y^{(i)}\right) \forall\, x \in \mathbb{R}^n, y \in \mathbb{Z}^m\,, \tag{17}$$

Solution. The global minimum is located at $\hat{x} = (0{,}0,\dots,0)^t$, and $\hat{y} = (0{,}0,\dots,0)^t$, and a value of $f(\hat{x},\hat{y}) = 0$.

A.4 W problem

We propose a multimodal asymmetric function defined by

$$\min_{x\in\mathbb{R}^n, y\in\mathbb{Z}^m} f(x) + g(y), \tag{18}$$

where

$$f(x) = \sum_{i=1}^{n}\left(\left(x^{(i)}/a\right)^8 + 2 - {x^{(i)}}^2\right) - \prod_{i=1}^{n} e^{-\left(\frac{x^{(i)}-b}{c}\right)^2}, \forall\, x^{(i)} \in \mathbb{R}\,, \tag{19}$$

$$g(y) = \sum_{i=1}^{m}\left(\left(y^{(i)}/a\right)^8 + 2 - {y^{(i)}}^2\right) - \prod_{i=1}^{m} e^{-\left(\frac{y^{(i)}-b}{c}\right)^2}, \forall\, y \in \mathbb{Z}\,, \tag{20}$$

$$a = 4\sqrt{2}\,,\ b = -8,\ c = 1/2. \tag{21}$$

Solution. The global minimum is located at $\hat{x} = (-8,-8,\dots,-8)^t$, and $\hat{y} = (-8,-8,\dots,-8)^t$, and a value of $f(\hat{x},\hat{y}) = -46(n+m) - 2$.

A.3 Mixed integer Tang problem

We show a modified version in the mixed integer numerical field of the Tang function, which is formulated in the real field by Shi and Ólafsson (2000).

$$\min_{x\in\mathbb{R}^n, y\in\mathbb{Z}^m} f(x,y), \tag{22}$$

subject to

$$3 \le x^{(i)} \le 13, \forall i \in \{1,\dots n\}, \tag{23}$$

$$3 \le y^{(j)} \le 13, \forall j \in \{1,\dots m\}, \tag{24}$$

where

$$f(x,y) = \sum_{i=1}^{n}\sin\left(x^{(i)}\right) + \sin\left(\tfrac{2}{3}x^{(i)}\right) + \sum_{j=1}^{m}\sin\left(y^{(j)}\right) + \sin\left(\tfrac{2}{3}y^{(j)}\right) \forall\, x \in \mathbb{R}^n, y \in \mathbb{Z}^m\,, \tag{25}$$

Solution. The global minimum is located at $\hat{x} = (5.3714, 5.3714, \dots, 5.3714)^t$, and $\hat{y} = (5,5, \dots ,5)^t$, and a value of $f(\hat{x}, \hat{y}) = -1.2159n - 1.1495m$.

References

Audet, C., and J. E. Dennis Jr., Pattern Search Algorithms for Mixed Variable Programming: *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 573-594, 2001.

Brea, E., Extensiones del método de Nelder Mead a problemas de variables enteras y enteras mixtas, Caracas, Universidad Central de Venezuela, p. xiv, 138, 2009

Brea, E. Una extensión del método de Nelder Mead a problemas de optimización no lineales enteros mixtos. *Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería,* vol. 29, no.3, pp. 163-174, 2013.

Brea, E. On the convergence of the Mixed Integer Randomized Pattern Search Algorithm. Available: https://www.researchgate.net/publication/265164860, March 2015.

Brea, E. On the Performance of the Mixed Integer Randomized Pattern Search Algorithm. *Proceedings of the XIII Congreso Internacional de Métodos Numéricos en Ingeniería y Ciencias Aplicadas* (Cimenics 2016), Caracas, July 2016.

Floudas, C. A. *Nonlinear and mixed-integer optimization: fundamentals and applications*. New York: Oxford University Press, 1995.

Fu, M., Marcus, S., & Hu, J. Model-Based Randomized Methods for Global Optimization. *Proceedings of 17th International Symposium on Mathematical Theory of Networks and Systems*, Kyoto, Japan. July 24-28, 2006. Available: http://www-ics.acs.i.kyoto-u.ac.jp/~mtns06/MTNS2006.pdf, August 2006.

Mazalov, V. V. *Mathematical game theory and applications* (1ed.). Chichester, United Kingdom: John Wiley and Sons, Ltd., 2014.

Shi, L., & Ólafsson, S. Nested Partitions method for global optimization. *Operations Research,* vol. 48, no. 3, pp. 390-407, 2000.

Yang, Y., Rubio, F., Scutari, G., and Palomar, D., 2012, Multi-portfolio Optimization: A Potential Game Approach, *in* Jain, R., and Kannan, R., editors, Game Theory for Networks: Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering, Springer Berlin Heidelberg, pp. 182-189, 2012

Ebert Brea is a Full Professor of the School of Electrical Engineering at the Central University of Venezuela, and the School of Industrial Engineering of the Andrés Bello Catholic University. He received his PhD from the Department of Mathematics of the University of Southampton, a MSc in Operational Research and Electrical Engineer degree, both from the Faculty of Engineering of the Central University of Venezuela. His research interests include Monte Carlo simulation, development of simulation models of discrete event dynamic systems, optimization algorithm design and optimization by simulation. He currently is Head of Department of Electronics, Computing and Control of the Central University of Venezuela, Coordinator of Engineering Science Doctoral Programmer of the Central University of Venezuela, and held the position of Director of the School of Electrical Engineering of the Central University of Venezuela. He has been awarded three times for his contributions to the Central University of Venezuela with: the Medal of Honor; the Plaque of Honor; and the Tie of Honor "Dr José María Vargas".