

# A Polynomial Time Algorithm for the Intermodal Hub Location Problem

Tao Wu, Ameya Dhaygude, Barnali Bhattacharjee

Advanced Analytics Department

Dow Chemical

Midland, MI 48642, USA

[twu1@dow.com](mailto:twu1@dow.com) ([danielwu9999@gmail.com](mailto:danielwu9999@gmail.com)), [Dhaygude@dow.com](mailto:Dhaygude@dow.com), [BBhattacharjee@dow.com](mailto:BBhattacharjee@dow.com)

## Abstract

We propose an intermodal hub location problem with market selection. The problem has a given number of markets and the revenue associated with each market. Each market has a predetermined demand, and the selected market demand has to be shipped from the origin to the destination via a hub-and-spoke system. The overall objective is to maximize the overall margins that are equal to the total revenue of the selected markets minus the total costs including the operating costs of intermodal hubs, transportation costs, and outsourcing costs. We propose a mixed integer programming model to formulate the problem without capacity constraints (IHL-MU). For the problem, we propose a Lagrangian relaxation based polynomial time algorithm that exploits the rich structure of the Lagrangian subproblems to obtain the objective values of the problems and their objective bounds efficiently. Specifically, we develop a polynomial time algorithm to solve the Lagrangian subproblems. The proposed method can obtain near-optimal solutions for large test instances, while the commercial software package (CPLEX 12.6) cannot achieve feasible solutions.

## Keywords

Hub location, mixed integer programming, polynomial time, heuristics, Lagrangian relaxation.

## 1. Introduction

Hubs serve as a set of fully interconnected facilities that function as centralized handling, connecting, consolidating, and switching points for flows between origins and destinations. Hubs are the centers of a network design where a large number of direct movements can be substituted with fewer, indirect connections. The hub systems save and simplify the network configuration, management, and operating costs. The cost rate for the inter-hub transportation is less than the shipment between a hub and an origin/destination because of economies of scale such that hubs can save the total shipment and connection costs. The hub location problems aim to assign hubs to serve as transportation and consolidating points for flow movements between specified origins and destinations. The objectives of the problems are to minimize total associated costs that may include transportation costs between hubs and origins/destinations and hub operating costs.

Mathematical models have been proposed for the hub location problems, including the single allocation single-hub location problem (O’Kelly (1987)), the single allocation p-hub location problem (O’Kelly (1987)), p-hub center location problem (Ernst et al. (2009)), and p-hub covering location problem (Kara and Tansel (2003)). Most of these problems are challenging and computationally intractable by explicit enumeration due to a combinatorially increasing number of potential solution alternatives. This attracts an extensive amount of research being conducted for finding effective solution methodologies, including exact methods (e.g., Benders decomposition (Contreras et al. (2011a)), branch-and-bound (Mayer and Wagner (2002)), and branch-and-price (Contreras et al. (2011b)) and heuristics (e.g., ant colony optimization (Meyer et al. (2009)), Tabu search (Abyazi-Sani and Ghanbari (2016)), informative optimization (He et al. (2015)), hybrid algorithms (Zhang et al. (2017)), and genetic algorithm (Damgacioglu et al. (2015))). The exact methods are efficient at solving small and uncapacitated problems. However, they are limited on thoroughly searching all possible branches and finding an optimal solution under a reasonable amount of computational time when the problem size becomes large. Heuristics and meta-heuristics are not guaranteed to achieve proven optimal solutions but aim to find good feasible solutions in a reasonable amount of computational time. Farahani et al. (2013) have given a detailed review on the heuristics and meta-heuristics for the hub location problems. For other detailed reviews on the hub location problems, please refer to Campbell and O’Kelly (2012).

In this paper, we propose an intermodal hub location problem with market selection (IHL-MS). This problem has applications for the freight transport companies providing truck delivery services in a hub-and-spoke network. The problem aims to locate fully interconnected facilities called intermodal hubs served as trans-shipment and switching points for moving flows between specified origins and destinations. The transportation via intermodal hubs is usually more cost-effective than the direct origin-to-destination transportation, as intermodal hubs have equipment specifically designed for transferring containers and truck trailers among different modes that are connected via rails, while rail transportation becomes more efficient than truck transportation when distances are over a certain threshold and results in savings in operating costs.

The IHL-MS problem possesses of three connected movements: Two local ones that involve truck movements and an intermediate one that involves rail movement. There are a number of customers' orders to be selected, and each order is associated with a given revenue yield. The overall objective is to maximize the overall margins that are equal to the total revenue of the selected customers minus the total costs of the system, which balance the operating costs of intermodal hubs, transportation costs of flows, and outsourcing costs when some of the flows are preferred to be transported by other resources. For the IHL-MS problem, we make the following contributions in this paper.

- We propose an MIP model to formulate the problems without capacity constraints (IHL-MU). The model with a large size is complicated to solve for the commercial CPLEX solver.
- A Lagrangian relaxation based polynomial time algorithm is developed for solving the IHL-MU problem, for which we discover the special structure of the Lagrangian subproblems and propose an efficient polynomial time algorithm to solve them. Further, we propose a polynomial algorithm to quickly generate objective values of the original problems at each iteration of the Lagrangian relaxation. This Lagrangian relaxation based polynomial time algorithm is a specially designed heuristic so it has critical differences from other Lagrangian relaxation heuristics proposed in the literature, e.g., Aykin (1994), Lee et al. (1996), Contreras et al. (2009), Contreras et al. (2010), Zhang et al. (2013), and Wu et al. (2013).
- We tested the polynomial time algorithm on problems with small, medium, and large sizes and compared the heuristic with the branch-and-bound method embedded in a commercial solver (CPLEX 12.6). The results show that the CPLEX solver can obtain optimal solutions for the problems with small and medium sizes but cannot even achieve a feasible solution for the large-size problems. The Lagrangian relaxation based polynomial time algorithm can produce optimal or near-optimal solutions for the small- and medium-size problems and can obtain near-optimal solutions for the large-size test problems.

The remainder of this paper is organized as follows. In Section 2, we propose mathematical formulations for IHL-MU. In Section 3, we describe the Lagrangian relaxation based polynomial time algorithm for the IHL-MU problem. Section 4 shows extensive computational results and comparisons of the proposed polynomial time algorithm with the CPLEX solver. Finally, we conclude with future directions in Section 5.

## 2. Mathematical Formulation for the IHL-MS Problem

The IHL-MS problem studied in this paper is similar to the one presented in He et al. (2015). However, we additionally consider the market selection and revenue yields into the problem. The IHL-MS problem has a set ( $F$ ) of markets and the revenue ( $ry_f$ ) associated with each market. Each market has a predetermined demand ( $w_f$ ), and there is a possibility to either satisfy or reject the demand of a market depending on whether the market is profitable. The overall objective is to maximize the overall margins that are equal to the total revenue of the selected markets minus the total costs including the operating costs ( $oc_h$ ) of intermodal hubs, transportation costs ( $tc_{hl}$  and  $hc_{h_1h_2}$ ), and outsourcing costs ( $vc_f$ ). To present the problem formulations, we define the following notations:

### Sets:

- $H$  Set of intermodal hubs, indexed by  $h$ .
- $L$  Set of origins and destinations, indexed by  $l$ .
- $F$  Set of flows (markets are also referred to as flows in this paper), indexed by  $f$ .
- $A$  Set of arcs,  $h_1h_2$ ,  $lh$ ,  $hl$  and  $l_1l_2 \in A$ .

### Parameters:

- $O_f$  Origin of flow  $f$ .
- $D_f$  Destination of flow  $f$ .

$w_f$	Amount of flow $f$ .
$tc_{hl}$	Per-amount transportation cost between intermodal hub $h$ and origin/destination $l$ .
$\alpha$	A discount factor for the inter-hub transportation.
$hc_{h_1h_2}$	Per-amount transportation cost between intermodal hubs $h_1$ and $h_2$ .
$oc_h$	Operating cost of intermodal hub $h$ .
$vc_f$	Per-amount transportation cost for flow $f$ if the flow is outsourced to other operations.
$ry_f$	Revenue yields for serving flow $f$ .

**Variables:**

$z_{fh_1h_2}$	Proportion of flow $f$ transported through intermodal hubs $h_1$ and $h_2$ .
$v_f$	Proportion of flow $f$ outsourced to other operations.
$y_h$	Binary hub open variables: $y_h=1$ if intermodal hub $h$ is open, 0, otherwise.
$s_f$	Binary market (flow) selection variables: $s_f=1$ if flow $f$ is selected, 0, otherwise.

**IHL-MU:**

$$\max \left\{ \sum_{f \in F} ry_f \cdot s_f - \sum_{f \in F} \sum_{l_1=O_f} \sum_{h_1 \in H} \sum_{h_2 \in H} \sum_{l_2=D_f} (tc_{l_1h_1} + \alpha \cdot hc_{h_1h_2} + tc_{h_2l_2}) \cdot w_f \cdot z_{fh_1h_2} - \sum_{h \in H} oc_h \cdot y_h - \sum_{f \in F} vc_f \cdot w_f \cdot v_f \right\} \quad (1)$$

subject to:

$$\sum_{\substack{h_1, h_2 \in A \\ h_2 \in H}} z_{fh_1h_2} \leq y_{h_1}, \quad \forall f \in F, h_1 \in H \quad (2)$$

$$\sum_{\substack{h_1, h_2 \in A \\ h_1 \in H}} z_{fh_1h_2} \leq y_{h_2}, \quad \forall f \in F, h_2 \in H \quad (3)$$

$$v_f + \sum_{h_1 \in H} \sum_{\substack{h_1, h_2 \in A \\ h_2 \in H}} z_{fh_1h_2} = s_f, \quad \forall f \in F \quad (4)$$

$$z_{fh_1h_2} = 0, \quad \forall l_1 \in O_f, l_1h_1 \notin A \quad (5)$$

$$z_{fh_1h_2} = 0, \quad \forall l_2 \in D_f, h_2l_2 \notin A \quad (6)$$

$$0 \leq z_{fh_1h_2} \leq 1, v_f \geq 0, \quad \forall h_1, h_2 \in H, f \in F \quad (7)$$

$$s_f \in \{0,1\}, \quad \forall f \in F \quad (8)$$

$$y_h \in \{0,1\}, \quad \forall h \in H \quad (9)$$

Constraints (2) and (3) guarantee that an intermodal hub must be open if there are any flows moving through the hub. Constraints (4) ensure that all flows must be shipped either via inter-hub transportation or via other outsourcing operations. Constraints (5) and (6) assure that flow transportation is disallowed between two unconnected intermodal hubs. Constraints (7)-(9) enforce integrality and non-negativity requirements for various variables.

### 3. A Lagrangian Relaxation Based Polynomial Time Algorithm for IHL-MU

We relax the constraints (2) and (3) in the Lagrangian relaxation method. We let  $\psi_{fh}$  and  $\theta_{fh}$  be vectors of Lagrange multipliers for the constraints (2) and (3), respectively. After relaxing the constraints and dualizing the terms into the objective function, we obtain the following Lagrangian subproblem  $[Z(\psi, \theta)]$ :

$$Z(\psi, \theta) = \max \left\{ \sum_{f \in F} r y_f \cdot s_f - \sum_{f \in F} \sum_{h_1 \in H} \sum_{h_2 \in H}^{h_1 h_2 \in A} \left( \sum_{l_1 \in O_f}^{l_1 h_1 \in A} t c_{l_1 h_1} \cdot w_f + \alpha \cdot h c_{h_1 h_2} \cdot w_f + \sum_{l_2 \in D_f}^{h_2 l_2 \in A} t c_{h_2 l_2} \cdot w_f + \psi_{f h_1} + \theta_{f h_2} \right) \cdot z_{f h_1 h_2} - \sum_{h \in H} \left( o c_h - \sum_{f \in F} \psi_{f h} - \sum_{f \in F} \theta_{f h} \right) \cdot y_h - \sum_{f \in F} v c_f \cdot w_f \cdot v_f \right\}$$

subject to: (4) – (9)

### 3.1. Solutions of the Lagrangian Subproblem $Z(\psi, \theta)$

The hub location variables and the other variables are disjoined in the Lagrangian subproblem. Herein, the Lagrangian subproblem  $Z(\psi, \theta)$  can be separated to two smaller problems: one considers the hub location variables, and the other considers the remaining decision variables. Furthermore, the second problem can be further separated to various smaller problems by flow  $f$ ,  $\forall f \in F$ . To describe the problems, we let  $\pi_{f h_1 h_2} = \sum_{l_1 \in O_f}^{l_1 h_1 \in A} t c_{l_1 h_1} \cdot w_f + \alpha \cdot h c_{h_1 h_2} \cdot w_f + \sum_{l_2 \in D_f}^{h_2 l_2 \in A} t c_{h_2 l_2} \cdot w_f + \psi_{f h_1} + \theta_{f h_2}$  and  $\tau_h = \sum_{f \in F} \psi_{f h} + \sum_{f \in F} \theta_{f h} - o c_h$ . We define the problems as  $Z_1(\psi, \theta)$  and  $Z_2^f(\psi, \theta)$  given as follows:

$$Z_1(\psi, \theta) = \max \left\{ \sum_{h \in H} \tau_h \cdot y_h \mid y_h \in \{0, 1\}, \forall h \in H \right\}$$

$$Z_2^f(\psi, \theta) = \max \left\{ r y_f \cdot s_f - \sum_{h_1 \in H} \sum_{h_2 \in H}^{h_1 h_2 \in A} \pi_{f h_1 h_2} \cdot z_{f h_1 h_2} - v c_f \cdot w_f \cdot v_f \right\}, \quad \forall f \in F$$

subject to:

$$v_f + \sum_{h_1 \in H} \sum_{h_2 \in H}^{h_1 h_2 \in A} z_{f h_1 h_2} = s_f,$$

$$z_{f h_1 h_2} = 0, \quad \forall l_1 \in O_f, l_1 h_1 \notin A,$$

$$z_{f h_1 h_2} = 0, \quad \forall l_2 \in D_f, h_2 l_2 \notin A,$$

$$0 \leq z_{f h_1 h_2} \leq 1, v_f \geq 0, \quad \forall h_1, h_2 \in H,$$

$$s_f \in \{0, 1\}.$$

We explore the structures of problems  $Z_1(\psi, \theta)$  and  $Z_2^f(\psi, \theta)$  and develop a polynomial algorithm to solve them optimally. To describe the algorithm, we let  $\tilde{A}_f$  be the set of all hub-hub connections for flow  $f$ ,  $\hat{A}_f$  be the route that requires the smallest costs for flow  $f$ , and  $\pi_f^{min}$  be the smallest cost among all routes that can serve flow  $f$ , for which we have  $\tilde{A}_f = \{h_1 h_2 \mid h_1 \in H, h_2 \in H, h_1 h_2 \in A, l_1 \in O_f, l_1 h_1 \in A, l_2 \in D_f, h_2 l_2 \in A\}$ ,  $\pi_f^{min} = \arg \min_{h_1 h_2 \in \tilde{A}_f} \pi_{f h_1 h_2}$ , and  $\hat{A}_f = \{h_1 h_2 \mid \pi_{f h_1 h_2} = \pi_f^{min}\}$ . If there are multiple routes that have the same minimum cost, we arbitrarily pick one of them as  $\hat{A}_f$ . Further, we define  $\hat{y}_h$  and  $\hat{Z}_1(\psi, \theta)$  as the respective optimal solution and objective value of  $Z_1(\psi, \theta)$  and  $(\hat{s}_f, \hat{z}_{f h_1 h_2}, \hat{v}_f)$  and  $\hat{Z}_2^f(\psi, \theta)$  as the respective optimal solution and objective value of  $Z_2^f(\psi, \theta)$ . It is obvious that  $(\hat{y}_h, \hat{s}_f, \hat{z}_{f h_1 h_2}, \hat{v}_f)$  is an optimal solution of  $Z(\psi, \theta)$ , and we have the below proposition.

**Proposition 3.1** The optimal objective value  $\hat{Z}(\psi, \theta)$  of  $Z(\psi, \theta)$  is equal to  $\hat{Z}_1(\psi, \theta) + \sum_{f \in F} \hat{Z}_2^f(\psi, \theta)$ .

The optimal solution can be easily obtained for problem  $Z_1(\psi, \theta)$  for which we have  $\hat{y}_h = 1$  if  $\tau_h > 0$  and  $\hat{y}_h = 0$  if  $\tau_h < 0, \forall h \in H$ . We note that the optimal values of  $\hat{y}_h$  can be either one or zero if  $\tau_h = 0$ . For this case, we take zero as the optimal value. The procedure for obtaining the optimal solution of  $Z_2^f(\psi, \theta)$  is described in Algorithm 1:

The logic of the algorithm is straightforward. A flow is not served if its revenue yield is smaller than the minimum cost  $\min(\pi_f^{min}, v c_f \cdot w_f)$ . Otherwise, we satisfy the flow demand. When a flow is served, whether the flow is served using an internal route with the minimum cost  $\pi_f^{min}$  or using the external services is dependent on

whichever one that has a smaller cost. We note that, for the scenarios  $ry_f = \min(\pi_f^{min}, vc_f \cdot w_f)$  or  $\pi_f^{min} < vc_f \cdot w_f$ , there are possibly several optimal solutions that lead to the same optimal objective, we have arbitrarily taken one of them in the algorithm.

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**Algorithm 1:** Solving the Lagrangian Subproblem  $Z_2^f(\psi, \theta), \forall f \in F$

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if  $ry_f \leq \min(\pi_f^{min}, vc_f \cdot w_f)$  then
     $(\hat{s}_f, \hat{z}_{fh_1h_2}, \hat{v}_f) := (0, 0, 0), \forall h_1h_2 \in \tilde{A}_f$ 
else if  $ry_f > \min(\pi_f^{min}, vc_f \cdot w_f)$  and  $\pi_f^{min} < vc_f \cdot w_f$  then
     $(\hat{s}_f, \hat{z}_{fh_1h_2}, \hat{v}_f) := (1, 1, 0), \forall h_1h_2 \in \hat{A}_f$ 
     $\hat{z}_{fh_1h_2} := 0, \forall h_1h_2 \notin \hat{A}_f, h_1h_2 \in \tilde{A}_f$ 
else
     $(\hat{s}_f, \hat{z}_{fh_1h_2}, \hat{v}_f) := (1, 0, 1), \forall h_1h_2 \in \tilde{A}_f$ 
end
end
end
Output:  $(\hat{s}_f, \hat{z}_{fh_1h_2}, \hat{v}_f)$  and  $\hat{Z}_2^f(\psi, \theta)$ 

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**Proposition 3.2** *The method described as above is a polynomial time algorithm with a degree of 1, and its computational complexity is  $O(|F| \cdot |A|)$ .*

*Proof.* The method described as above has three parts: i) solve problem  $Z_1(\psi, \theta)$  optimally, ii) calculate  $\pi_f^{min}$ , and iii) solve problems  $Z_2^f(\psi, \theta)$  optimally using Algorithm 1. It is straightforward to determine that the respective complexities of parts i, ii, and iii are  $O(|H|)$ ,  $O(|F| \cdot |A|)$ , and  $O(|F| \cdot |A|)$ . Herein, the total computational complexity is  $O(|F| \cdot |A|)$ , and the method is a polynomial time algorithm with a degree of 1.

### 3.2. The Description of the Lagrangian Relaxation Based Polynomial Time Algorithm

For any choice of the Lagrange multipliers  $(\psi, \theta)$ , the solution of the Lagrangian subproblems provides an upper bound to the problems. To obtain the best possible upper bound, we have to solve the Lagrangian dual:

$$\mathcal{L}_{IHL-MU} = \min_{\psi, \theta} Z(\psi, \theta)$$

A subgradient direction can be calculated to update  $(\psi, \theta)$  iteratively and then the Lagrangian dual problem can be solved using a specially designed algorithm. We note that the achieved solution for the Lagrangian dual problem may not be directly feasible for the original IHL-MU problem. Herein, we apply a fast heuristic to construct a feasible solution to the original problem at each iteration of the subgradient procedure and update the solution progressively. Before describing the polynomial time algorithm, we provide the following notations:

$\epsilon$	The iteration number.
$\tilde{Z}$	The incumbent objective value for the IHL-MU problem.
$(\psi^\epsilon, \theta^\epsilon)$	The dual multipliers at iteration $\epsilon$ .
$\lambda$	The movement size at the subgradient procedure.
$(\hat{y}^\epsilon, \hat{s}^\epsilon, \hat{z}^\epsilon, \hat{v}^\epsilon)$	An optimal solution of the Lagrangian subproblem $Z(\psi^\epsilon, \theta^\epsilon)$ at iteration $\epsilon$ .
$\hat{Z}(\psi^\epsilon, \theta^\epsilon)$	The optimal objective value of the Lagrangian subproblem $Z(\psi^\epsilon, \theta^\epsilon)$ at iteration $\epsilon$ .
$\omega$	A predefined parameter.

**The Lagrangian Relaxation Based Polynomial Time Algorithm:**

**Step I - Initialization:** Set  $\tilde{Z} = -\infty$ ,  $\epsilon = 1$ ,  $\psi^\epsilon = 0$ , and  $\theta^\epsilon = 0$ . Go to Step II.

**Step II – Obtain solutions of the Lagrangian subproblem:** We solve the Lagrangian subproblem  $Z(\psi^\epsilon, \theta^\epsilon)$  using the method described in Section 3.1 to obtain the optimal objective value  $\hat{Z}(\psi^\epsilon, \theta^\epsilon)$  and an optimal solution  $(\hat{y}^\epsilon, \hat{s}^\epsilon, \hat{z}^\epsilon, \hat{v}^\epsilon)$ . Go to Step III.

**Step III - Update the Lagrangian dual multipliers  $(\psi, \theta)$ :** Obtain the Lagrange multipliers  $(\psi^{\epsilon+1}, \theta^{\epsilon+1})$  using the subgradient method stated in Section 3.3 to. Go to Step IV.

**Step IV – Achieve solutions of the original IHL-MU problem:** We construct a feasible solution of the original IHL-MU problem using the optimal solution  $(\hat{y}^\epsilon, \hat{s}^\epsilon, \hat{z}^\epsilon, \hat{v}^\epsilon)$  of the Lagrangian subproblem. Let  $\hat{y}_n^\epsilon = 1$  if

$\sum_{f \in F} (\sum_{h_2 \in H} \sum_{h_1 \in A} \hat{z}_{fh_1h_2}^\epsilon + \sum_{h_1 \in H} \sum_{h_2 \in A} \hat{z}_{fh_1h_2}^\epsilon) > 0$  and  $\hat{y}_h^\epsilon = 0$  otherwise.  $(\hat{y}^\epsilon, \hat{s}^\epsilon, \hat{z}^\epsilon, \hat{v}^\epsilon)$  is a feasible solution of the original IHL-MU problem, for which the corresponding objective value of IHL-MU can be easily calculated. We update  $\hat{Z}$  whenever a better objective value is found. Go to Step V.

**Step V - Termination:** The stopping criteria used for the method are: i) if the gap between  $\hat{Z}$  and  $\hat{Z}$  is less than a tolerance, or ii) if the maximum iteration number or the maximum computational time is reached, or iii)  $\hat{Z}$  and  $\hat{Z}$  have not improved for a specified number of iterations. We stop the algorithm if the termination criteria are met. Otherwise, we set  $\epsilon = \epsilon + 1$  and go to Step II.

At Step IV, we generate a feasible solution based on the solution  $(\hat{y}^\epsilon, \hat{s}^\epsilon, \hat{z}^\epsilon, \hat{v}^\epsilon)$  of the Lagrangian subproblems. It is easy to identify that the solution procedure is a polynomial time algorithm with a degree of 1. The derived solution provides a lower bound for the optimal integer solution of the IHL-MU problem that directly impacts the convergence of the subgradient method. It is important that it is achieved quickly because the solution of the Lagrangian dual involves many iterations.

### 3.3. The Subgradient Method for IHL-MU

We compute the subgradient directions  $\xi_{fh_1}^\epsilon$  and  $\gamma_{fh_2}^\epsilon$  using the optimal solution values  $(\hat{y}^\epsilon, \hat{s}_f, \hat{z}_{fh_1h_2}, \hat{v}_f)$  of the Lagrangian subproblem:

$$\xi_{fh_1}^\epsilon = \sum_{\substack{h_1h_2 \in A \\ h_2 \in H \\ h_1h_2 \in A}} \hat{z}_{fh_1h_2}^\epsilon - \hat{y}_{h_1}^\epsilon \quad (10)$$

$$\gamma_{fh_2}^\epsilon = \sum_{\substack{h_1h_2 \in A \\ h_1 \in H}} \hat{z}_{fh_1h_2}^\epsilon - \hat{y}_{h_2}^\epsilon \quad (11)$$

The step size  $\lambda$  is calculated using scheme (12) at the first iteration, and it is calculated by scheme (13) at other iterations.

$$\lambda = \frac{w \cdot \hat{Z}(\psi^\epsilon, \theta^\epsilon)}{\sum_{f \in F} \sum_{h_1 \in H} (\xi_{fh_1}^\epsilon)^2 + \sum_{f \in F} \sum_{h_2 \in H} (\gamma_{fh_2}^\epsilon)^2} \quad (12)$$

$$\lambda = \frac{w \cdot (\hat{Z}(\psi^\epsilon, \theta^\epsilon) - \hat{Z})}{\sum_{f \in F} \sum_{h_1 \in H} (\xi_{fh_1}^\epsilon)^2 + \sum_{f \in F} \sum_{h_2 \in H} (\gamma_{fh_2}^\epsilon)^2} \quad (13)$$

Parameter  $\omega$  is decreased whenever the objective value of the Lagrangian subproblems has failed to improve after a specified number of steps. The Lagrangian multipliers  $\psi_{fh}$  and  $\theta_{fh}$  are updated using subgradient-based optimization schemes (14) and (15).

$$\psi_{fh_1}^{\epsilon+1} = \max\{0, \psi_{fh_1}^\epsilon + \lambda \cdot \xi_{fh_1}^\epsilon\} \quad (14)$$

$$\theta_{fh_2}^{\epsilon+1} = \max\{0, \theta_{fh_2}^\epsilon + \lambda \cdot \gamma_{fh_2}^\epsilon\} \quad (15)$$

## 4. Computational Results

As the proposed problem was not studied in the literature, we generated three sets of test instances using different parameter settings for computational tests, with their problem sizes and names given in Table 1.

Table 1. Set Names and Problem Sizes for Computational Tests

Problem type	Set ID	Set name	$ L $	$ H $	$ F $
IHL-MU	1	MU202020	20	20	20
	2	MU802080	80	20	80
	3	MU200200400	200	200	400

We assume origins/destinations and hubs are located in a square area described by the Euclidean Coordinate System with four corners' coordinates given as (0, 0), (800, 0), (0, 800), and (800, 800). To generate the test instances, we apply a full factorial design method to generate problems with various properties. In the experimental design, there are three levels of flow capacities and two levels of four parameters (transportation costs, capacities, hub open costs, and revenue yields). The detailed setting of parameters is given as follows:

- $tc_{hl}$  This parameter has two settings denoted by small and large that respectively indicate the per-amount transportation cost between intermodal hub  $h$  and origin/destination  $l$  has a uniform distribution in  $\{1, 3\} \cdot [0.002, 0.003] \cdot dist_{hl}$ , where  $dist_{hl}$  is the Euclidean distance between  $h$  and  $l$ .
- $\alpha$  This parameter has a uniform distribution in  $[0.3, 0.5]$ .
- $w_f$  The flow amount is distributed in  $ceil(rand(0, 3)) \cdot 1000$ .
- $hc_{h_1h_2}$  This parameter has two such settings denoted by small and large that respectively indicate the per-amount transportation cost between intermodal hubs  $h_1$  and  $h_2$  has a uniform distribution in  $\{1, 3\} \cdot \alpha \cdot [0.002, 0.003] \cdot dist_{h_1h_2}$ . We note that the coefficient  $\{1, 3\}$  used for setting  $tc_{hl}$  and  $hc_{h_1h_2}$  is the same for a test instance.
- $oc_h$  This parameter has two settings denoted by small and large that indicate the open cost of intermodal hubs has a uniform distribution in  $\{1, 3\} \cdot [Y, 2 \cdot Y]$  where the parameter  $Y$  is defined as  $\frac{\sum_{f \in F} w_f}{|F|} \cdot \left( \frac{\sum_{h_1 \in H} \sum_{h_2 \in H} hc_{h_1h_2}}{|H|^2} + \frac{\sum_{h \in H} \sum_{l \in L} tc_{hl}}{|H| \cdot |L|} \right)$ .
- $vc_f$  The per-unit amount and per-unit distance outsource cost has a uniform distribution in  $[0.02, 0.03]$ .
- $ry_f$  This parameter has two such settings denoted by small and large that indicate the revenue yields for serving flow  $f$  has a uniform distribution in  $\{0.5, 1.5\} \cdot [\Theta(f), 5 \cdot \Theta(f)]$  where  $\Theta(f) = w_f \cdot \left( \frac{\sum_{h_1 \in H} \sum_{h_2 \in H} hc_{h_1h_2}}{|H|^2} + \frac{2 \cdot \sum_{h \in H} \sum_{l \in L} tc_{hl}}{|H| \cdot |L|} \right) + \frac{\sum_{h \in H} oc_h}{4 \cdot |F|}$ .

According to the experimental design, there are 8 test instances in each problem set. The above experimental design is similar to the one applied in He et al. (2015) but with additional considerations of revenue yields. Because there are no existing methods proposed for solving the IHL-MU problem in the literature, we compare the proposed heuristics with a commercial solver, IBM ILOG CPLEX 12.6. All approaches were programmed using GAMS 24.3 with a computing capacity (Intel Pentium 4, 2.5 GHz processor and 8.0 GB RAM).

We compare the Lagrangian relaxation based polynomial time algorithm (LRPT) with the CPLEX solver under the default setting using Sets 1-3. The computational time limits assigned for the two methods are 200 seconds for the small-size and medium-size problems in Sets 1-2 and 3,600 seconds for the large-size problems in Set 3. For LRPT, parameters  $\omega$  is initially set to 2, which is decreased by a rate of 1.01 whenever the best objective value of the Lagrangian subproblems has not been improved for 20 iterations. Further, we note that, in early iterations, the step size  $\lambda$  calculated using the equations (12) and (13) may be huge, which causes issues for the subgradient procedure. The computational results showed that the performance of LRPT is enhanced if we set the maximal step size to be 100. Besides the maximal time limits mentioned above, the additional termination criteria include: the solution quality (i.e., the gap between the objective value  $\tilde{Z}$  and the upper bound  $\hat{Z}(\psi^\epsilon, \theta^\epsilon)$ ) does not improve more than 0.2 after 200 iterations, the algorithm exceeds 3,000 iterations, and the gap between  $\tilde{Z}$  and  $\hat{Z}(\psi^\epsilon, \theta^\epsilon)$  is less than 0.001.

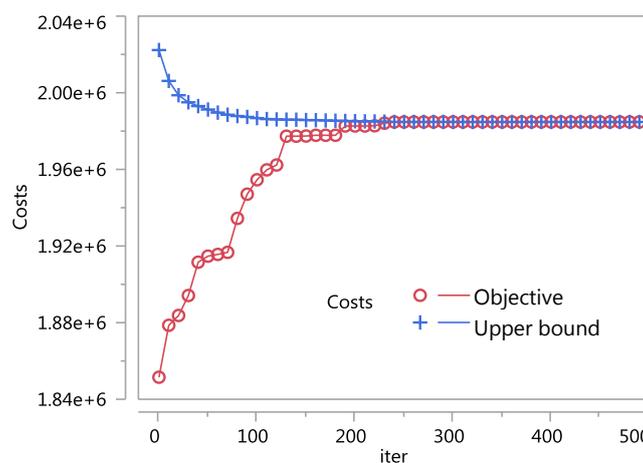


Figure 1 The Computational Results of LRPT for MU202020

Table 2. Comparisons of Computational Results of LRPT and CPLEX for MU802080

Setting	LRPT (3 s)				LRPT (200 s)				CPLEX
	OBJ	BND	GP	OPT-G	OBJ	BND	GP	OPT-G	OPT
<b>Shipping cost</b>									
Small	3,671,080	3,793,764	4.37%	2.97%	3,741,225	3,754,556	0.47%	0.42%	3,753,018
Large	11,013,241	11,381,291	4.37%	2.97%	11,224,068	11,262,961	0.46%	0.41%	11,259,054
<b>Hub open cost</b>									
Small	7,089,364	7,212,047	2.36%	1.33%	7,157,521	7,157,521	0.00%	0.00%	7,157,521
Large	7,594,958	7,963,008	6.38%	4.61%	7,807,771	7,859,995	0.94%	0.83%	7,854,550
<b>Revenue yield</b>									
Small	3,345,611	3,590,978	6.67%	4.55%	3,486,097	3,512,249	0.72%	0.64%	3,509,486
Large	11,338,710	11,584,077	2.07%	1.39%	11,479,196	11,505,267	0.22%	0.19%	11,502,585
Average	7,342,161	7,587,528	4.37%	2.97%	7,482,646	7,508,758	0.47%	0.42%	7,506,036

Table 3. Comparisons of Computational Results of LRPT and CPLEX for MU200200400

Setting	LRPT			CPLEX	
	OBJ	BND	GP	OBJ	GP
<b>Shipping cost</b>					
Small	19,864,928	20,527,337	4.27%	–	–
Large	59,594,785	61,600,597	4.30%	–	–
<b>Hub open cost</b>					
Small	36,843,548	37,457,190	2.27%	–	–
Large	42,616,165	44,670,744	6.30%	–	–
<b>Revenue yield</b>					
Small	18,237,091	19,571,478	6.52%	–	–
Large	61,222,622	62,556,456	2.04%	–	–
Average	39,729,857	41,063,967	4.28%	–	–

Figure 1 shows the average performance of  $\bar{Z}$  and  $\min_{\epsilon} \hat{Z}(\psi^{\epsilon}, \theta^{\epsilon})$  for all eight small-size problems, indicating that LRPT can achieve their optimal solutions using an average of 250 iterations and 5.5 seconds. The computational results also show that the CPLEX solver can achieve their optimal solutions using an average of 1.7 seconds, meaning that CPLEX has an advantage of solving the small-size problems optimally. However, the computational results indicate that LRPT can obtain a feasible solution point much quicker than CPLEX. Comparatively, LRPT can get a feasible solution point in 0.03 seconds while CPLEX takes more than 1 second to get a feasible solution point.

The computational results of LRPT and CPLEX for the medium- and large-size problems are given in Tables 2 and 3, where OPT represents the optimal objective values obtained by CPLEX, and OBJ, BND, GP, and OPT-G represent the respective objective values, upper bounds, and optimality gaps achieved by LRPT. We calculate GP using the difference between the upper bound and the best objective value divided by the upper bound. OPT-G indicates the gap between the best objective value achieved by a method and the optimal objective value, which is calculated using the difference between the optimum and the best objective value divided by the best objective value. All these notations have the same meanings in the following tables.

The computational results show that CPLEX can obtain the optimal solutions for all eight medium-size problems using an average of 8 seconds, while LRPT takes an average of 147 seconds to achieve objective values that are 0.42% away from the optimal objective values, and LRPT achieves the optimal objectives of four problems using an average of 95 seconds. Although CPLEX can outperform LRPT for solving the medium-size problems optimally, LRPT can obtain feasible objective values that are at most 3% away from the optimal objectives in 3 seconds, while CPLEX cannot achieve any feasible solutions in such a short time. Additionally, CPLEX loses

advantages to LRPT for the large-size problems, for which CPLEX cannot achieve a feasible solution in one hour, but LRPT can obtain both feasible objective values and upper bounds that yield an average of 4.28% optimality gaps. The parameter settings of the hub open costs and revenue yields have impacts on the solution qualities of LRPT, with bigger optimality gaps achieved for the problems with larger hub open costs and smaller revenue yields. The setting of the shipping cost has a less noticeable impact on the solution quality.

## 5. Conclusions and Future Research

This paper proposes an MIP model to formulate the intermodal hub location problem with market selection. A Lagrangian relaxation based polynomial time algorithm is developed for solving the hub location problem. The rich structure of the Lagrangian subproblems is explored and an efficient polynomial algorithm is developed to solve the subproblems. We also developed a polynomial algorithm to quickly generate objective values of the original problems at each iteration of the Lagrangian relaxation. The proposed algorithm is compared with CPLEX using the test problems with various sizes. The results show that CPLEX can obtain optimal solutions for the small-size problems quicker than the proposed methods. However, the proposed methods can obtain solutions with good qualities for the large-size test problems, for which CPLEX cannot get feasible solutions using the same computing time. For the future research, we are interested in extending the IHL-MU problem with considerations of flow and hub capacities and developing analytics-driven metaheuristics (e.g., Liang et al. (2015) and Wu et al. (2018)) for solving the problems.

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## Biographies

**Tao Wu** has been working as a Data Scientist in the Advanced Analytics group at The Dow Chemical Company since 2014. His research interests are in the arena of building mathematical, analytical, and statistical models, and developing algorithms, methodologies, and theories for solving real industrial problems. His work has been applied in price optimization and revenue management, manufacturing, production planning, supply chain management, marketing and finance, energy management, and logistics. He has been serving as an Associate Editor for the International Journal of Systems Science. He has published more than 20 refereed international journal papers, and the work has appeared in *INFORMS Journal on Computing*, *Transportation Research Part B.*, *Transportation Research Part E.*, *Journal of Global Optimization*, *Annals of Operations Research*, *European Journal of Operational Research*, *Omega*, *International Journal of Production Economics*, *Computers & Operations Research*, *International Journal of Production Research*, and *IEEE Trans. on Automation Science and Engineering*, etc. Before joining Dow, he worked at Apollo Group Inc. and General Motors as Operations Research Scientist and Research Engineer, respectively. Earlier, he was granted with a Ph.D. degree in Industrial and Systems Engineering in 2010 and a Master degree in computer science in 2009 from The University of Wisconsin-Madison.

**Ameya Dhaygude** has been working as a Data Scientist in the Advanced Analytics group at The Dow Chemical Company since 2011. Ameya has expertise in operations research and machine learning methods. He is leading development of scalable analytical solutions for supply chain and purchasing processes at Dow. Ameya has successfully implemented operations research and optimization algorithms which has enabled multi-million dollar value for Dow. For example, he has implemented Mixed Integer Linear Programs (MILPs) to optimize supply chain networks, applied stochastic program to hedge price risk of raw materials, developed econometric time series models to forecast demand and prices, used decision tree to classify invoice data, etc. Prior to joining Dow, Ameya received his Master's in Industrial Engineering and Management from the Oklahoma State University in 2010. He worked as a research assistant for Dr. Baski Balasundaram during his graduate studies. Ameya's research work and Master's thesis focused on developing metaheuristic algorithms to solve the chance constrained minimum spanning  $k$ -core problem.

**Barnali Bhattacharjee** has been working as a Data Scientist and Analytics Deployment Lead in the Advanced Analytics group at The Dow Chemical Company since early 2018. Her work focuses on developing innovative and data-driven solutions for a wide array of business problems, with an emphasis on marketing, sales, supply chain, and procurement, and has expertise in predictive modeling, machine learning, and strategic decision making. Prior to joining Dow, Barnali received her Master's in Statistics from Indian Institute of Technology Kanpur in 2010, another Master's in Business Analytics in 2017 from Michigan State University, and worked for 5 years in Retail & Corporate Banking leading and developing data science solutions.