Mixtures-based Value at Risk Estimates of Financial Stocks

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Abstract

Value at risk (VaR) and Conditional VaR (CVaR) are two common measures of risk that are related to the loss distribution. It is generally believed that if the true loss distribution is heavy-tailed, as compared to the normal, then the risk is higher. We show that in general this is not the case. We derive formulas for VaR and CVaR for mixtures and show that there are instances where the normality assumption overestimates (and the mixture distribution underestimates) the observed market risk. We show examples using market data from different financial firms to confirm our conclusions.

Keywords
Value at Risk, mixture normal distribution, risk management

1. Introduction

Value at risk is a simple probability measure of the loss of an investment. When associated with a percentile p of the loss distribution, the p-Value at risk (VaR_p) is defined as the smallest loss that may be observed, with probability p, over a specified time period.
The Conditional value at risk was proposed by Artzner et al. (2003) as an extension of Value at risk. When used with a percentile p the p-Conditional Value at risk (CVaR_p) is the expected loss beyond VaR_p.
The returns of a large number of financial time series show heavy tails. When returns are assumed normal the risk measures such as VaR_p and CVaR_p may underestimate the true risk for large p. To verify if this is the case, we fit a mixture and a normal distribution to daily returns of financial stocks with moderate and large kurtosis. We then estimated and compare their risk values.
All computations are performed using *R* (R Core Team, 2014). The reader not familiar with *R* may find a good introduction in Venables and Smith (2009).
It is found that if p and kurtosis are very large, the mixture provides the best risk measures, but if p and kurtosis are moderately large (say, $p > 0.95$ and $\kappa < 10$) then the normal distribution of returns provides the best risk measures.

2. Mixtures

In finance applications the distribution of returns shows heavy tails (i.e., very large negative returns are possible). To accurately model these distributions, random variables with heavy-tailed distributions are considered. These are distributions with large kurtosis. For instance the *t*, logistic or Pareto distributions. In this paper a mixture of normal random variables to model daily returns with zero mean, is considered.
Let $R_1, R_2, \ldots, R_n$ be a set of *iid* daily returns with mixture pdf given by

$$f(r) = \alpha f_1(r) + (1 - \alpha) f_2(r) \tag{1}$$

where the mixture components are given by f_i, the density of a normal with variance σ_i^2, for $i = 1,2$ and the coefficients α and $(1 - \alpha)$ are the associated weights. We assume that the variables have zero mean (which is usually the case for daily returns). We also assume that the mixture component 2 is more volatile, $\sigma_2 > \sigma_1$. The mixture (1)

is able to model two market scenarios, one with low volatility, σ_1, and the other with high volatility, σ_2 (and therefore more risk), with probabilities α and $(1-\alpha)$, respectively.
To find VaR_p and CVaR_p the quantiles of a mixture distribution are required. It can be shown that the *p*-quantile of (1) is the solution of

$$p = \alpha\Phi\left(\frac{r_p}{\sigma_1}\right) + (1-\alpha)\Phi\left(\frac{r_p}{\sigma_2}\right) \tag{2}$$

where $\Phi()$ is the standard normal distribution function.
Note that for a normal return with zero mean and variance σ^2 the *p*-quantile is given by

$$r_p = \sigma z_p \tag{3}$$

where z_p is the *p*-quantile of the standard normal. It can be verified that (3) is a special case of (2) if $\alpha = 1$ and $\sigma = \sigma_1$.

3. Value at Risk and Conditional Value at Risk

There are several ways to measure the risk of financial investments. The variance and the kurtosis are two of the first measures used for risk estimation. The variance is a measure of how much, on average, the return deviates from its mean. The kurtosis measures how large (or *fat*) the tails of a distribution are. In both cases large values indicate that the investment is risky. The variance and kurtosis for a mixture can be found as follows.
For mixture (1) the variance is given by

$$\sigma^2 = \alpha{\sigma_1}^2 + (1-\alpha){\sigma_2}^2 \tag{4}$$

and the kurtosis is given by

$$\kappa = 3\left[\alpha\left(\frac{{\sigma_1}^2}{\sigma^2}\right)^2 + (1-\alpha)\left(\frac{{\sigma_2}^2}{\sigma^2}\right)^2\right] \tag{5}$$

where σ^2 is given by (4). It is not difficult to see that when $\alpha = 1$ and $\sigma_1 = \sigma$ then the kurtosis is equal to 3.0. That is, the kurtosis of normal random variables is 3.0 always.
The variance and kurtosis measure the overall deviance of the distribution around its tails. But to measure investment risk, negative deviance is more important since it is the risk of a negative return or loss. Two risk measures that focus on these deviations are value at risk and conditional value at risk.
The *p*-Value at Risk (VaR_p) is defined as the *p*-quantile of the loss distribution. It is the smallest loss with a 100*p*% probability to be observed. Value at risk can be found from the percentile of daily returns. If *I* is the investment and *R* is the daily return, the resulting loss *L* is given by

$$L = -IR \tag{6}$$

The *p*-Value at Risk of investment *I* is therefore

$$\mathrm{VaR}_p = -Ir_p \tag{7}$$

and can be found using (2) for a mixture and (3) for normal returns.
The *p*-conditional value at risk (CVaR_p) is the expected loss beyond VaR_p. It is defined as $E\left[L|L > \mathrm{VaR}_p\right]$. Intuitively the *p*-conditional value at risk answers the question, given that a loss is to be observed with probability *p*, how large could it be, on average? It can be shown that the *p*-conditional value at risk of (1) is

$$\mathrm{CVaR}_p = \frac{1}{1-p}\left[\alpha\sigma_1\phi\left(\frac{r_p}{\sigma_1}\right) + (1-\alpha)\sigma_2\phi\left(\frac{r_p}{\sigma_2}\right)\right] \tag{8}$$

and the p-conditional value at risk of a normal return with volatility σ is

$$\text{CVaR}_p = \frac{1}{1-p}\left[\sigma\phi\left(r_p\right)\right] \tag{9}$$

where $\phi()$ is the standard normal density function.

Table 1. Parameters of a two-component normal mixture

Stock	Component	Weight	Volatility
1	C_1	0.7	0.01
1	C_2	0.3	0.04
2	C_1	0.9	0.01
2	C_2	0.1	0.10

4. Comparing the risk of two stocks

To see how VaR and CVaR compare when returns show low or high kurtosis, we consider two stocks with the parameters shown in Table 1.
Using (4) and (5) it can be found that the volatility and kurtosis of Stock 1 are equal to 0.02345 and 7.686 respectively. Those for Stock 2 are 0.033 and 25.273 respectively. That is, Stock 1 returns are moderately heavy-tailed while Stock 2 returns are highly heavy-tailed. Stock 2 shows larger volatility than Stock 1 and therefore it is more risky.
We compare their value at risk and conditional value at risk with that of normal returns with the same mean and variances varying p from 0.85 to 0.99. The result is shown in Figure 1. Curves in the top two plots are for Stock 1, and bottom two plots are for Stock 2. The leftmost (rightmost) plots show VaR (CVaR) as a function of p. The dashed curve shows values corresponding to normal returns.
It is clear that in all cases the curves intersect. That is, no model (normal or mixture of normals) reports that a stock is more risky for all p values.
When returns are normal, value at risk is larger than that of a mixture. Only when p is close to 1, VaR curve of the mixture exceeds that of normal returns. That is, unless we consider p close to 1, normal returns report larger VaR than returns from a mixture. Moreover, as kurtosis increases, and therefore the Stock is more risky, Value at Risk from normal returns is larger unless p is close to 1. It seems that returns from a mixture, are not reporting the risk of a heavy-tailed distribution as clear as returns from a normal distribution.
To compare the Conditional value at risk we focus on the two rightmost plots. For moderately heavy-tailed distribution, the mixture reports the Stock risk larger than the normal returns. Only when large heavy-tailed distribution with very large kurtosis value are considered, the normal returns outperform mixture returns and report larger Conditional value at risk.
As we will see in the following section, most returns from financial data show moderately large kurtosis values, in the range from 5 to 10. Thus, Conditional value at risk from mixtures is a better measure of stock risk than that of normal returns. If Value at risk is preferred the opposite is true though.

5. Comparison using Financial data

In this section we consider daily returns from two US companies with moderate and large kurtosis values. We use *R* library `moments` (Komsta and Novomestky, 2015) to find the kurtosis of the daily returns. We select Ford Motor Co. (Ford) and H&R Block Inc. (HRB) since it was found that the kurtosis of their 2016 daily returns was equal to 6.374 (moderate) and 20.67 (large), respectively.
For each company we estimate the historical (sample) VaR_p and CVaR_p values as follows. Mixture (1) is fitted to the daily returns using *R* library `mixtools` (Young et al., 2017). Function `normalmixEM` applies the standard EM algorithm of McLachlan and Peel (2000) to find estimates of parameters α, σ, and, σ_2, and (2) and (8) are used to find VaR_p and CVaR_p per dollar invested (I = 1). Also (4) is used to find the mixture variance σ^2. Finally, we find VaR_p and CVaR_p for a

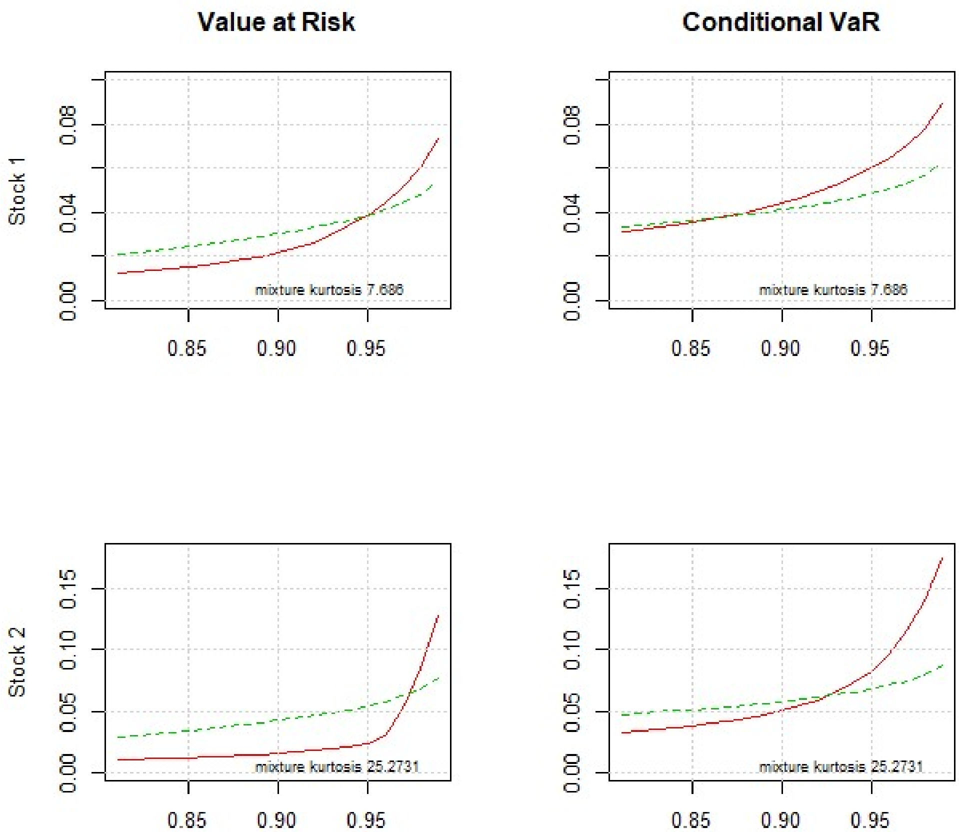

Figure 1. VaR and Conditional VaR of two stocks

Table 2. Parameters of the fitted normal mixtures

Estimate	Ford	HRB
kurtosis κ	6.3741	20.670
weight α	0.7292	0.9606
volatility σ_3	0.0105	0.0137
volatility σ_2	0.0262	0.0927
volatility σ	0.0164	0.0228

normal distribution with the same variance σ^2 as that found for the mixture. The estimated 2016 parameter values for Ford and HRB are shown in Table 2.

We let $0.80 \leq p \leq 0.99$ to report the curves shown in Figure 2. These curves resemble those from Figure 1 but now with real data. The top (bottom) plots show the resulting curves for Ford (HRB). The continuous curves show risk values for the mixture, the dots show historical (sample) VaR_p and $CVaR_p$ values, and the dashed curves the values from the fitted normal distribution.

It can be seen that for returns with moderate kurtosis (such as Ford) the mixture fits best the VaR historical values unless p is large $p > 0.95$, and the fitted normal distribution fits best the CVaR historical values unless p is small $p < 0.86$. For practical purposes, if $p > 0.95$ use the fitted normal distribution to fit the returns to obtain risk estimates.

For returns with large kurtosis (such as HRB) the mixture fits best the VaR historical values unless p is very large $p \geq 0.99$, and it also fits best the CVaR historical values unless $p \geq 0.95$. For practical purposes, if $p \geq 0.99$ use the fitted normal distribution to fit the returns to obtain risk estimates. Otherwise use the mixture fit.

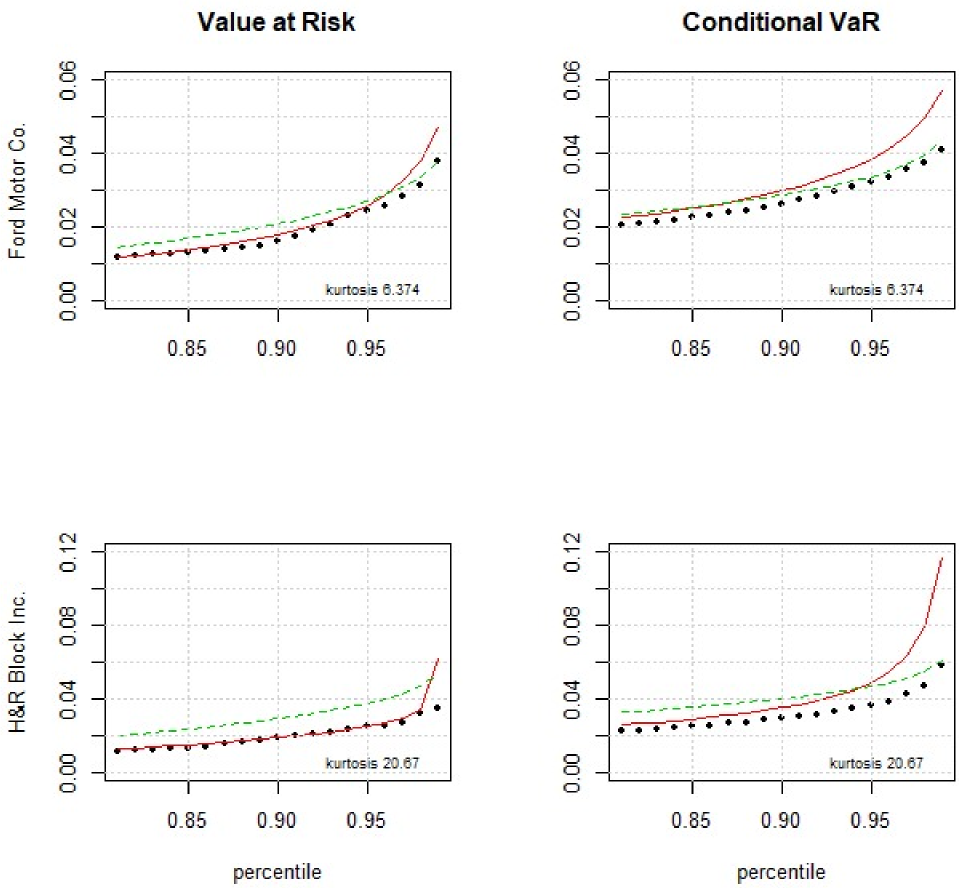

Figure 2. Ford and HRB 2016 risk estimates

Conclusions

It is believed that if the true distribution is heavy-tailed then the risk is higher. To verify this statement we compare VaR and CVaR risk estimates based on historical data, under the assumption of normal and a mixture of normal returns. We derive formulas for these estimates and use them to find estimates for stocks with large and extremely large kurtosis returns.

We find that the p-VaR from a mixture of normal estimates the observed VaR very accurately. However if p is close to one, there are two scenarios. If returns show moderate excess kurtosis than normal VaR estimates are better. And if returns show extreme large kurtosis, then both the normal and the mixture estimates fail to accurately estimate historical VaR. On the other hand, Conditional VaR under the normal assumption estimates the historical CVaR much accurately when p is very close to one.

Therefore, for practical purposes, to estimate CVaR we suggest fitting returns with a normal distribution, while for VaR, the mixture of normals shows a better performance.

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Biographies

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