Mathematical Modelling and Analysis of Human Arm as a Triple Pendulum System using Euler-Lagragian Model

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Abstract

This paper attempts to model the human arm as a dynamical Triple pendulum system. The equation of Motion of the human arm was obtained using Euler-Lagrange equation. The resulted second order differential equation was solved analytically. Simulated results were presented with the aid of a computer software - Maple. It was observed that the angular displacement values of the three segments are directly proportional to their respective angular acceleration, which is in line with what is available in the literature. However, the novelty of this work is in the modelling and analysis of human arm motion as a multiple pendulum system. Generally, the longer the segments of the human arm the longer it takes to swing back-and-forth, and the fewer back-and-forth swings there are in a second.

Keywords: Mathematical Modelling, Human Arm, Triple Pendulum System, Euler-Lagrange.

1. Introduction

Human parts move when involved in activities such as walking, running, dancing, jumping and so on. The proper movement of these parts of the body results in good body balance. Proper functioning of human arms results in a good human activity. Using Mechanics and other Mathematical concepts for human body part motion modelling and analysis is constantly expanding and becoming very important in body mechanics via the application of Newtonian mechanics to the human skeletal systems [1]. Agarana et al, considered the movement of human arm during dance. He pointed out the importance balanced arm movement in a good dance. In dancing, the arm locomotion is one of the most complicated motions of a human body [1,2,3].
Human body or part of it always strives to maintain balance. So, during any activity the balancing of human arm ensures good and sustained position, at least for a considerable long period of time. Body mechanics involves the coordinated effort of muscles, bones, and the nervous system to maintain balance, posture, and alignment during moving, transferring, and positioning a body. Proper body mechanics allows individuals to carry out human activities without excessive use of energy and helps prevent injuries [3,4]. During human activities, a balance of the body is maintained through body mechanics. When a vertical line falls from the Centre of gravity through the wide base of support, body balance is achieved, otherwise the body will lose balance [4]. Balance in this sense means an ability to maintain the line of gravity (vertical line from Centre of mass) of a body within the base of support with minimal postural sway[4]. Human arm comprises the upper arm, the lower arm and the hand. Each of these is represented by one of the three simple

pendulums that make up the triple pendulum system, A triple pendulum can be referred to as a combination of three simple pendulums [5]. A simple pendulum is one with the pivot at the top and the mass at the bottom [5,6,7]. Dissipative and driven forces can be accounted for by splitting the external forces into a sum of potential and non-potential forces [8,9]. The Lagrangian [11] is a time derivative mathematical function of the generalized coordinates that contains the information about the dynamics of the system. Lagrangian mechanics is ideal for systems with conservative forces and for bypassing constraint forces in any coordinate system [11]. The dynamic analysis of the different segments of the human arm during activities was analysed in this paper. The study is very relevant to body mechanics.

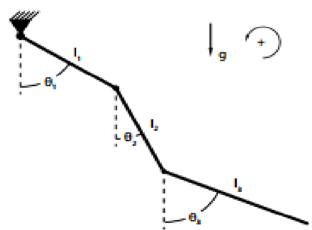

Fig.2. Schematic of Triple pendulum [1]

2. Modelling Human Arm

For modelling, the human arm can be simulated as three links; where upper arm is considered as the first link that is jointed with elbow. Also, the lower part of the arm is assumed as the second link that is connected to the wrist. The palm is considered as the third link. To simply model of the human arm the upper-link rotation angle from a vertical position is denoted by θ_1 and the corresponding rotation angle for the lower-links by θ_2 and θ_3 respectively. The length of link between shoulder and elbow joint is shown by l_1 and length of link between elbow and wrist joint is shown by l_2. Friction and other dissipating forces are assumed negligible. Figure 1 shows the schematic of triple pendulum representing the three segments of a typical human arm in its downward position. The modelling of the dynamical system results in second order differential equation [14].

2.1 Mathematical Model Formulation

A triple pendulum consists of one pendulum attached to another, then to another. This is an example of dynamical system which can exhibit chaotic behaviour. Consider human arm modelled as a triple bob pendulum with masses m_1, m_2 and m_3 attached by rigid massless wire of lengths l_1, $l_2$and l_3. The angles the wire make with the vertical are represented as θ_1, θ_2 *and* θ_3 respectively.

Following the work of Agarana; the acceleration due to gravity is g and the positions of the bobs are given respectively as:

(x_1,y_1), (x_2,y_2) and (x_3,y_3).

where

$$x_1 = l_1 sin\theta_1 \quad (1)$$

$$x_2 = l_1 sin\theta_1 + l_2 sin\theta_2 \quad (2)$$

$$x_3 = l_1 sin\theta_1 + l_2 sin\theta_2 + l_3 sin\theta_3 \quad (3)$$

$$y_1 = -l_1 \cos\theta_1 \quad (4)$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \quad (5)$$

$$y_3 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 - l_3 \cos\theta_3 \quad (6)$$

It is assumed in this paper that the values of θ_1, θ_2 *and* θ_3 ranges from 0 to 90 degrees. This implies that none of the segments of the human arm, during any activity, should make more than 90 degrees with the vertical.

Following Agarana's work, the potential energy of the system is then given as

$$V = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 \quad (7)$$

while the kinetic energy of the system is given as

$$T = \frac{1}{2}\mathrm{m}_1\mathrm{v}_1^2 + \frac{1}{2}\mathrm{m}_2\mathrm{v}_2^2 + \frac{1}{2}\mathrm{m}_3\mathrm{v}_3^2 \quad (8)$$

V and T can be rewritten respectively as

$$V = -(m_1 + m_2 + m_3) g l_1 \cos\theta_1 - (m_2 + m_3) g l_2 \cos\theta_2 - m_3 g l_3 \cos\theta_3 \quad (9)$$

and

$$T = \frac{1}{2}\mathrm{m}_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}\mathrm{m}_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \frac{1}{2}\mathrm{m}_3\left(\dot{x}_3^2 + \dot{y}_3^2\right) \quad (10)$$

After substituting the differentials of $x_1, x_2, x_3, y_1, y_2, y_3$ into equation (13) and carry out some rearrangement leads to:

$$T = \frac{1}{2}[(m_1 + m_2 + m_3) l_1^2 \dot{\theta}_1^2 + (m_2 + m_3) l_2^2 \dot{\theta}_2^2 + m_3 l_3^2 \dot{\theta}_3^2]$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)$$
$$+ 2 m_3 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 (\cos\theta_1 \cos\theta_3) + m_3 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 (\cos\theta_2 \cos\theta_3 + \sin\theta_2 \cos\theta_3) \quad (11)$$

The Lagrangian L, is given as:

$$L = T - V \quad (12)$$

$$L = \frac{1}{2}[(m_1 + m_2 + m_3)l_1^2\dot{\theta}_1^2 + (m_2 + m_3)l_2^2\dot{\theta}_2^2 + m_3 l_3^2\dot{\theta}_3^2]$$
$$+m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) + 2m_3 l_1 l_3\dot{\theta}_1\dot{\theta}_3(\cos\theta_1\cos\theta_3)$$
$$+m_3 l_2 l_3\dot{\theta}_2\dot{\theta}_3(\cos\theta_2\cos\theta_3 + \sin\theta_2\cos\theta_3) + (m_1 + m_2 + m_3)gl_1\cos\theta_1$$
$$+(m_2 + m_3)gl_2\cos\theta_2 + m_3 gl_3\cos\theta_3 \quad (13)$$

The Euler – Lagrangian equation is given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0 \quad (14)$$

Evaluating the Euler – Langrangian equation for $\dot{\theta}_1, \dot{\theta}_2$ and $\dot{\theta}_3$, for non-stationary values respectively gives

$$(m_1 + m_2 + m_3)(l_1\ddot{\theta}_1 - g\sin\theta_1) + m_3 l_2[\ddot{\theta}_2\cos\theta_1\cos\theta_2$$
$$+\ddot{\theta}_2\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1 + \sin\theta_1\sin\theta_2$$
$$-\dot{\theta}_2\cos\theta_1\sin\theta_2 - \dot{\theta}_2\cos\theta_2\sin\theta_1 + \dot{\theta}_2\cos\theta_1\cos\theta_2$$
$$-\dot{\theta}_2\sin\theta_1\sin\theta_2] + 2l_2\dot{\theta}_2[m_2\ddot{\theta}_3\cos\theta_1\cos\theta_3$$
$$+m_3\dot{\theta}_1\dot{\theta}_3\cos\theta_3\sin\theta_1 - m_2\dot{\theta}_3\cos\theta_1\sin\theta_3$$
$$-m_2\dot{\theta}_3\sin\theta_1\cos\theta_3] = Q_1 \quad (15)$$

$$(m_2 + m_3)(l_2\ddot{\theta}_2 + g\sin\theta_2) + m_3 l_1[\ddot{\theta}_1\cos\theta_1\cos\theta_2$$
$$+\ddot{\theta}_1\cos\theta_1\sin\theta_2 - \dot{\theta}_1\cos\theta_1\sin\theta_2 - \dot{\theta}_1\sin\theta_1\cos\theta_2$$
$$+\dot{\theta}_1\cos\theta_1\cos\theta_2 - \dot{\theta}_1\sin\theta_2\sin\theta_1$$
$$+\dot{\theta}_1\dot{\theta}_2\cos\theta_1\sin\theta_2 - \dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2]$$
$$+m_3 l_3[\ddot{\theta}_3\cos\theta_2\cos\theta_3 + \ddot{\theta}_3\cos\theta_3\sin\theta_2$$
$$-\dot{\theta}_3\cos\theta_2\sin\theta_3 - \dot{\theta}_3\sin\theta_2\cos\theta_3$$
$$+\dot{\theta}_3\cos\theta_2\cos\theta_3 - \dot{\theta}_3\sin\theta_3\sin\theta_2$$
$$+\dot{\theta}_2\dot{\theta}_3\sin\theta_2\cos\theta_3 - \dot{\theta}_2\dot{\theta}_3\cos\theta_3\cos\theta_2] = Q_2 \quad (16)$$

$$m_3(l_3\ddot{\theta}_3 - g\sin\theta_3) + 2m_2 l_1(\ddot{\theta}_1\cos\theta_1\cos\theta_3$$
$$-\dot{\theta}_1\cos\theta_1\sin\theta_3 - \dot{\theta}_1\cos\theta_3\sin\theta_1)$$
$$+m_3 l_2(\ddot{\theta}_2\cos\theta_3\cos\theta_2 + \ddot{\theta}_2\sin\theta_2\cos\theta_3$$
$$+\dot{\theta}_2\dot{\theta}_3\cos\theta_2\sin\theta_3 + \dot{\theta}_2\dot{\theta}_3\sin\theta_2\sin\theta_3$$
$$+\dot{\theta}_2\cos\theta_2\cos\theta_3 - \sin\theta_2\sin\theta_3$$
$$-\cos\theta_2\sin\theta_3 - \cos\theta_3\sin\theta_2)$$
$$+2m_3 l_1\dot{\theta}_1\dot{\theta}_3\cos\theta_1\sin\theta_3 = Q_3 \quad (17)$$

Where Q_1, Q_2, Q_3 are the (non-conservative) generalised forces.

3. Model Solution

From figure 1 above;

$$Sin\theta_1 = \frac{x_1}{l_1}, Sin\theta_2 = \frac{x_2}{l_2}, Sin\theta_3 = \frac{x_3}{l_3}$$

$$\Rightarrow \theta_1 = Sin^{-1}\frac{x_1}{l_1}, \theta_2 = Sin^{-1}\frac{x_2}{l_2}, \theta_3 = Sin^{-1}\frac{x_3}{l_3}$$

$$\Rightarrow \theta_1 = Sin^{-1}\frac{x_1}{\sqrt{y_1^2 + x_1^2}}, \theta_2 = Sin^{-1}\frac{x_2}{\sqrt{y_2^2 + x_2^2}}, \theta_3 = Sin^{-1}\frac{x_3}{\sqrt{y_3^2 + x_3^2}}$$

(x_1,y_1)	(1,1)	(1,2)	(2,4)	(3,5)	(5,2)
θ_1	0.707	0.447	0.447	0.514	0.928
l_1	1.414	2.236	4.472	5.831	5.385
(x_2,y_2)	(7,4)	(8,6)	(10,7)	(10,9)	(11,11)
θ_2	0.868	0.800	0.819	0.743	0.707
l_2	8.062	10.000	12.207	13.454	15.556
(x_3,y_3)	(12,12)	(13,12)	(13,14)	(14,15)	(15, 15)
θ_3	0.707	0.735	0.680	0.682	0.707
l_3	16.971	17.692	19.105	20.518	21.213

Table 1 : The angular displacement, lengths of segments and different coordinates

In order to solve for the masses, m_1, m_2 and m_3, the values of the parameters in table 1 are substituted into equations (31),(32) and (33) to become:

$$(m_1 + m_2 + m_3)(-g\sin\theta_1) + m_3 l_2[\cos\theta_2 \sin\theta_1 + \sin\theta_1 \sin\theta_2] = Q_1 \quad (18)$$

$$(m_2 + m_3)(g\sin\theta_2) = Q_2 \quad (19)$$

$$m_3(-g\sin\theta_3) + 2m_2 l_1(-\cos\theta_2 \sin\theta_3 - \cos\theta_3 \sin\theta_2) = Q_3 \quad (20)$$

For the purpose of this research work let Q_1, Q_2, Q_3 be assigned respectively the following values; 3,2,1.

When the three masses are at the position (1,1), (7,4) and (12,12) respectively, the absolute values of the masses can be evaluated from the following equations:

$$-0.121m_1 - 0.121m_2 + 15.044m_3 = 3 \quad (21)$$
$$0.149m_2 + 0.149m_3 = 2 \quad (22)$$
$$0.121m_3 + 0.08m_2 = -1 \quad (23)$$

$m_1 = 6378.68$, $m_2 = 64.01$, $m_3 = 50.59$

Similarly, on position (2,4), (10,7) and (13,14), the absolute values of the masses are:

$m_1 = 26.78$, $m_2 = 11.5$, $m_3 = 26.1$

For position (1,2), (8,6) and (13,12), the absolute values of the masses are:

$m_1 = 4.94$, $m_2 = 22.91$, $m_3 = 37.2$

For position (3,5), (10,9) and (14,15), the absolute values of the masses are:

$m_1 = 8.46$, $m_2 = 6.35$, $m_3 = 7.29$

For position (5,2), (11,11) and (15,15), the absolute values of the masses are:

3.1 Periods and Frequency of different segments of the pendulum system

The period of the motion for a pendulum is how long it takes to swing back-and-forth, measured in seconds. the frequency of a pendulum is how many back-and-forth swings there are in a second, measured in hertz [12]. Frequency f is the reciprocal of the period t: Looking at the three segments as different simple pendulums joined together, the periods of each segment, using the parameters obtained above, are obtained as follows:

The period of a simple pendulum is given as [20]:

$$T = 2\pi\sqrt{\frac{l}{g}} \quad (24)$$

Where l is the length and g is the acceleration due to gravity.

The periods for the three segments with different lengths, assuming a fixed initial position of the masses at (3,5), (10,9), (14,15) respectively, are shown in tables 2, 3, and 4 respectively.

Length of 1st Segment	Length	T_1	F_1
l_1	**5.83**	**4.82**	**0.21**
$2l_1$	**11.66**	**6.85**	**0.15**
$3l_1$	**17.49**	**8.38**	**0.12**

$4l_1$	**23.32**	**11.39**	**0.09**
$5l_1$	**29.15**	**10.82**	**0.092**
$6l_1$	**34.98**	**11.87**	**0.084**

Table 2: The values of the periods and frequency of first segment at given lengths.

Length of 2nd Segment	**Value**	**T_2**	**F_2**
l_2	**13.45**	**7.36**	**0.14**
$2l_2$	**26.9**	**10.41**	**0.096**
$03l_2$	**40.35**	**12.74**	**0.078**
$4l_2$	**53.8**	**14.72**	**0.068**
$5l_2$	**67.25**	**16.46**	**0.061**
$6l_2$	**80.7**	**18.03**	**0.055**

Table 3: The values of the periods and frequency of second segment at given lengths

Length of 3rd Segment	**Value**	**T_3**	**F_3**
l_3	**20.52**	**10.74**	**0.093**
$2l_3$	**41.04**	**15.19**	**0.066**
$3l_3$	**61.56**	**18.60**	**0.054**
$4l_3$	**82.08**	**21.48**	**0.047**
$5l_3$	**102.6**	**24.02**	**0.042**
$6l_3$	**123.12**	**26.31**	**0.038**

Table 4: The values of the periods and frequency of third segment at given lengths

4. Results and Discussion

To maintain balance during human activities, the required masses at the end of each segment of the human arm and their positions were calculated analytically. The values revealed that the position of the arm segments at every point in time, the mass at the end of each segment and the length of the segment are all important in the body mechanics analysis. From tables 2, 3, and 4, it can be seen that there is a general positive correlation between the length of the human arm segments and period but a negative correlation with the frequency. It was also observed from the study that the masses required at the end of the segments of the human arm depends on the position of that segment at a point in time, as shown the result from equations 21, 22, and 23. There is also a positive correlation between the angular displacement, angular acceleration and angular acceleration

5. Conclusion

This paper analytically modelled the dynamics of human arm as a triple pendulum system in motion. The angular displacements were determined by the simulated positions of the three segments of the human arm. Each of the three simple pendulums that form the triple pendulum represents each of the three segments of the human arm, namely; the upper arm, the lower arm and the hand. With Euler - Lagrange equations, the equations of motion of the triple pendulum were obtained. The solution to these equations reveal the dynamics of the segments of human arm. The sensitivity and interrelationship of the parameters were studied. For a good body mechanics to be achieved, especially about the locomotion of human arm, the analytical results of this paper give a clue of how stability and balance of the whole human arm movement during an activity can be supported.

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