

Applying Labeling Algorithm to Improve Sensitivity Analysis of Right-Hand-Side Parameters in Transportation Problems

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Abstract

The right-hand side parameters of the transportation problem, such as supply or demand values, sometimes can be determined by managerial policy decisions, which evaluate the marginal values of changing the supply or the demand values. Parameters in the transportation problem are estimated in practice, herein, sensitivity analysis can provide the information to realize the perturbation range while a parameter changing, which might vary the shipping flows or the shipping patterns. The most well-known algorithm for solving the transportation problem is stepping stone method, which is performed in a transportation tableau. However, few papers investigate implementing sensitivity analysis by transportation tableau. This paper proposes two lemmas, and modifies the labeling procedures in a transportation tableau to obtain the larger perturbation range, considering the invariant changing rate of objective function value. Sensitivity analysis technique can be especially implemented under the circumstances that no *more-for-less paradox* and exists the multiple equivalence shadow prices of the transportation problem. Consequently, the results of sensitivity analysis can provide decision makers more practical information, and can reduce the computing efforts on obtaining the boarder perturbation ranges.

Keywords

Transportation problem; Labeling algorithm; Sensitivity analysis; Shipping pattern.

1. Introduction

The transportation problem (TP) is the classical network flow problem, which provides the shipping patterns with minimal total cost for decision makers. The TP has been widely applied not only in commodity transportation (Hiller and Lieberman 2010), but also in electrical power transmission (Aarvik and Randolph 1975).

The TP satisfies the values of supply and demand simultaneously, is to find the shipping pattern with minimal transportation cost among the network. A TP considers m origin points and n destination points, where origin i has a supply of s_i units of the specific commodity, and destination j requires a demand of d_j units. A TP can then be formulated as a linear programming (LP) with standard form as model (1).

$$\begin{aligned} \min \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^n x_{ij} = s_i \quad \text{for } i = 1, 2, \dots, m, \\ & \sum_{i=1}^m x_{ij} = d_j \quad \text{for } j = 1, 2, \dots, n, \\ & x_{ij} \geq 0 \quad \text{for all } i \text{ and } j. \end{aligned} \tag{1}$$

where x_{ij} is the number of units shipped along link (i, j) , c_{ij} is the unit transportation cost along link (i, j) . Further, we assume total supply equals total demand as shown in (2).

$$\sum_{i=1}^m s_i = \sum_{j=1}^n d_j \quad (2)$$

The TP is usually solved in a transportation tableau, as shown in Table 1. The cell (i, j) represents the flow variable x_{ij} , where $i= 1, 2, \dots, m$, and $j= 1, 2, \dots, n$. Cost coefficient c_{ij} is on the north-west corner of each cell. Dual variables or the shadow prices u_i and v_j are shown in the left and upper margins, respectively. Reduced cost $\lambda_{ij}= c_{ij}- u_i- v_j$ is also displayed in the parenthesis of each non-basic cell. The most well-known algorithm for solving the TP is the stepping stone (SS) method, which is based on the primal-dual simplex method to obtain an optimal basic solution with exactly $m+ n- 1$ basic cells on the transportation tableau (Bazaraa et al. 2010).

Table 1: Parameters and variables table for the TP

| | | Destinations | | | | | | |
|---------|-----|--------------|------------------------------|-------|------------------------------|-------|------------------------------|-------|
| | | 1 | ... | j | ... | n | | |
| | | v_1 | ... | v_j | ... | v_n | | |
| Origins | 1 | u_1 | c_{11} (λ_{11}) | ... | c_{1j} x_{1j} | ... | c_{1n} (λ_{1n}) | s_1 |
| | : | : | : | ... | : | ... | : | : |
| | i | u_i | c_{i1} (λ_{i1}) | ... | c_{ij} (λ_{ij}) | ... | c_{in} x_{in} | s_i |
| | : | : | : | ... | : | ... | : | : |
| | m | u_m | c_{m1} x_{m1} | ... | c_{mj} (λ_{mj}) | ... | c_{mn} (λ_{mn}) | s_m |
| | | | d_1 | ... | d_j | ... | d_n | |

The parameters of the TP, such as cost coefficients of the objective function, and right-hand-side (RHS) parameters of constraints, are just estimates for future. Therefore, sensitivity analysis is implemented after obtaining the optimal solutions. Three types sensitivity analysis (Koltai and Terlaky 2000; Hadighi and Terlaky 2006; Hadighi et al. 2007) are proposed and by invariant characters. Following their definitions, these perturbation ranges can also be applied for the TP as follows:

1. **Type I (Invariant Basis) range** is the largest perturbation range for Δs_a or for Δd_b , which preserves the implemented optimal basis \mathbf{B}^* but the optimal solution may be changed. Type I sensitivity analysis is just the conventional sensitivity analysis and is based on the non-degeneracy assumption of the optimal solution for accuracy.
2. **Type II (Invariant Support Set) range** is the largest perturbation range for Δs_a or for Δd_b , which preserves the implemented optimal support set \mathbf{P}^* . Thus, the perturbation may change the loads on the implemented optimal transportation routes but preserve the same shipping pattern.
3. **Type III (Invariant Optimal Partition) range** is the largest perturbation range for Δs_a or for Δd_b , within which the rate of change of the optimal function value is constant.

Sensitivity analysis of RHS parameters in the TP are mentioned in literatures. However, those papers discuss equally perturb the supply and demand values, and discuss about the *more-for-less paradox* (Adlakha and Kowalski 1998; Adlakha and Kowalski 2000; Deineko et al. 2003; Adlakha et al. 2007; Schrenk et al. 2011). *More-for-less paradox* is a phenomenon that ships more products based on the optimal shipping pattern, but the total transportation cost is less. *More-for-less paradox* also exists in linear programming (Arsham 2007; Lin 2010), which is due to negative shadow prices for maximizing problem.

An ordinary sensitivity analysis, perturbs a single parameter at a time, is the most widely used type of sensitivity analysis. The ordinary sensitivity analyses of cost coefficients in the TP are usually carried out by transportation tableau (Wu and Coppins 1981), but few papers apply to RHS parameters of the TP. The sensitivity analyses of RHS

parameters are obtained by matrix operations (Adlakha and Arsham 1998; Badra 2007). Doustdargholi *et al.* (2009) obtain the perturbation ranges by algebraically operation. Balinski and Gomory (1967) propose an alternative method with labeling procedures to solve the TP. Several papers extend the algorithm to obtain the type II and type III sensitivity analysis of cost coefficient for the fractional assignment problem (Lin 2011) and the fuzzy assignment problem (Lin *et al.* 2011).

Suppose that supply a is perturbed with the value of Δs_a , and demand b is perturbed with the value of Δd_b , the perturbed the TP can be formulated as models (3) and (4) respectively.

$$\begin{aligned}
\min \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^n x_{aj} = s_a \pm \Delta s_a, \\
& \sum_{j=1}^n x_{ij} = s_i \quad \text{for } i = 1, 2, \dots, a-1, a+1, \dots, m, \\
& \sum_{i=1}^m x_{ij} = d_j \quad \text{for } j = 1, 2, \dots, n, \\
& x_{ij} \geq 0 \quad \text{for all } i \text{ and } j.
\end{aligned} \tag{3}$$

$$\begin{aligned}
\min \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^m x_{ib} = d_b \pm \Delta d_b, \\
& \sum_{j=1}^n x_{ij} = s_i \quad \text{for } i = 1, 2, \dots, m, \\
& \sum_{i=1}^m x_{ij} = d_j \quad \text{for } j = 1, 2, \dots, b-1, b+1, \dots, n, \\
& x_{ij} \geq 0 \quad \text{for all } i \text{ and } j.
\end{aligned} \tag{4}$$

The conventional sensitivity analysis is to find the maximal perturbation range while the optimal basis invariance. However, the parameters, especially the RHS parameters, sometimes are set by managerial policy decisions which are according to the marginal values. Besides, if there are multiple minimal negative reduced costs in the dummy row or dummy column, the conventional sensitivity analysis enters one of these cells to the basis. The objective function value can further be reduced by entering another cell with minimal negative reduced cost. Herein, sensitivity analysis, provide the perturbation ranges with keeping maximal marginal value for each RHS parameter, is more important to decision makers. The changing rate invariant sensitivity analysis of RHS parameters and Type III sensitivity analysis are analog. However, sensitivity analysis in this paper adds a dummy row or dummy column, which is not consistent to the definition of type III sensitivity analysis. Furthermore, few papers implement sensitivity analysis in the transportation tableau. Therefore, the purpose of this paper is to develop an algorithm with labeling procedure for obtaining sensitivity analysis of right-hand-side parameters of the TP in a transportation tableau. The rest of this paper is organized as follows. In Section 2, two lemmas are provided for determining the upper bound or lower bound of the perturbation range. Besides, an additional lemma is derived for detecting the *more-for-less paradox* of the TP. In Section 3, we propose a labeling algorithm for determining the perturbation ranges. A numerical example is presented to demonstrate the proposed algorithms in Section 4. Section 5 concludes the study with a brief summary.

2. The Perturbation Ranges for Invariant Changing Rate of Objective Function Value

In order to obtain the perturbation ranges of changing rate invariant sensitivity analysis, we derive the auxiliary perturbed problem of changing rate invariant sensitivity analysis. The auxiliary perturbed problem can help to prove some characters as shown in lemmas 1 and 2 to improve the efficiency of the algorithm in section 3. Sensitivity

analysis is carried out after obtaining the optimal solution. Without loss of generality, we assume both the TP and the auxiliary perturbed problem have at least one feasible solution. The following theorem 1 of LP is well known and useful later.

Theorem 1. Let the primal problem be feasible. If the dual problem is infeasible, then the optimal value of the primal problem is unbounded.

The auxiliary perturbed problem of increasing supply value is derived as model (5).

$$\begin{aligned}
& \max \quad \Delta s_a \\
& \text{s.t.} \quad \sum_{j=1}^{n+1} x_{aj} - \Delta s_a = s_a, \\
& \quad \quad \sum_{i=1}^m x_{a,n+1} - \Delta s_a = 0, \\
& \quad \quad \sum_{j=1}^{n+1} x_{ij} = s_i \quad \text{for } i = 1, 2, \dots, a-1, a+1, \dots, m, \\
& \quad \quad \sum_{i=1}^m x_{ij} = d_j \quad \text{for } j = 1, 2, \dots, n, \\
& \quad \quad x_{ij} \geq 0 \quad \text{for all } i \text{ and } j, \Delta s_a \in R.
\end{aligned} \tag{5}$$

The auxiliary perturbed problem of increasing demand value is derived as model (6).

$$\begin{aligned}
& \max \quad \Delta d_b \\
& \text{s.t.} \quad \sum_{i=1}^{m+1} x_{ib} - \Delta d_b = d_b, \\
& \quad \quad \sum_{j=1}^n x_{m+1,j} - \Delta d_b = 0, \\
& \quad \quad \sum_{j=1}^n x_{ij} = s_i \quad \text{for } i = 1, 2, \dots, m, \\
& \quad \quad \sum_{i=1}^{m+1} x_{ij} = d_j \quad \text{for } j = 1, 2, \dots, b-1, b+1, \dots, n, \\
& \quad \quad x_{ij} \geq 0 \quad \text{for all } i \text{ and } j.
\end{aligned} \tag{6}$$

Lemma 1. If the dual variable u_a is the largest value among all u_i , $i=1,2,\dots,m$, then the upper bound of supply a is ∞ .

Proof:

Model (7) is the dual of model (5).

$$\begin{aligned}
& \min \quad \sum_{i=1}^m s_i u_i + \sum_{j=1}^{n+1} d_j v_j + \Delta s_a (u_a + v_{n+1}) \\
& \text{s.t.} \quad u_a + v_{n+1} \geq 0, \\
& \quad \quad -u_a - v_{n+1} = 1, \\
& \quad \quad u_i + v_j \geq 0, \quad i = 1, 2, a-1, a+1, \dots, m, \quad j = 1, 2, \dots, n+1, \\
& \quad \quad u_a + v_j \geq 0, \quad j = 1, 2, \dots, n, \\
& \quad \quad u_i, v_j \in R.
\end{aligned} \tag{7}$$

Since the constraint $-u_a - v_{n+1} = 1 \Rightarrow u_a + v_{n+1} = -1$ contradicts with the constraint $u_a + v_{n+1} \geq 0$, model (7) is infeasible, which imply that model (5) has unbounded solution. Thus the proof is completed.

Lemma 2. *If the dual variable v_b is the largest value among all $v_j, j=1,2,\dots,n$, then the upper bound of demand b is ∞ .*

Proof.

Model (8) is the dual of model (6).

$$\begin{aligned}
 & \min \sum_{i=1}^{m+1} s_i u_i + \sum_{j=1}^n d_j v_j + \Delta d_b (u_{m+1} + v_b) \\
 & \text{s.t. } -u_{m+1} - v_b = 1, \\
 & \quad u_{m+1} + v_b \geq 0, \\
 & \quad u_i + v_j \geq 0, \quad i = 1, 2, \dots, m+1, \quad j = 1, 2, \dots, b-1, b+1, \dots, n, \\
 & \quad u_i + v_b \geq 0, \quad i = 1, 2, \dots, m, \\
 & \quad u_i, v_j \in R.
 \end{aligned} \tag{8}$$

Since the constraint $-u_{m+1} - v_b = 1 \Rightarrow u_{m+1} v_b = -1$ contradicts with the constraint $u_{m+1} + v_b \geq 0$, model (8) is infeasible, implying that model (6) has unbounded solution. Thus the proof is completed.

In summary, lemmas 1 provides the perturbation ranges for the invariant changing rate of objective function value sensitivity analysis, if u_a is the maximal value among $u_i, i=1,\dots,m$. Furthermore, lemmas 2 provides the perturbation ranges, if v_b is the maximal value among $v_j, j=1,\dots,n$. In section 3, the above lemmas are implemented to reduce the calculation efforts for the algorithm.

3. Labeling Algorithm

In this section, we propose a labeling algorithm to determine the perturbation range of the invariant changing rate of objective function value for each Δs_a and Δd_b . We assume that the non-degenerated optimal basic solutions with no alternative solution are given. The proposed algorithm is illustrated and performed in a transportation tableau. Before demonstrating the procedure of the algorithm, a frequently using sub-procedure *Labeling procedure* is shown as follows. If the non-basic cell (k, l) enter the basis, the sub-procedure *Labeling procedure* is to determine the closed loop which is started form cell (k, l) .

Labeling procedure: Suppose (k, l) is the pivot cell, which can enter the basis with $\lambda_{kl} \leq 0$.

- (1) Label column l with (k) .
- (2) If column q is labeled, we label row I with (q) where $x_{iq} > 0$.
- (3) If row p is labeled, we label column j with (p) where $x_{pj} > 0$ or reduced cost $\lambda_{pj} = 0$. If column j has been labeled, we keep its original label.
- (4) Repeat (2) and (3) until further labeling is impossible.

The main procedure is demonstrated as follows. γ_s^l and γ_s^u are respectively denoted as the lower bound of a supply value and the upper bound of a supply value; γ_d^l and γ_d^u are respectively denoted as the lower bound of a demand value and the upper bound of a demand value.

Step 0: Initialization

- (1) Find the optimal solution x_{ij} , and dual variables u_i and v_j by SS. Derive the reduced cost λ_{ij} , for $i=1, 2, \dots, m$, and $j=1, 2, \dots, n$.
- (2) While the value of supply a is increasing, and u_a is the maximal value among $u_i, i=1, \dots, m$, stop the algorithm, and let $\gamma_s^u \leftarrow \infty$ by lemma 1.
- (3) While the value of demand b is increasing, and v_b is the maximal value among $v_j, j=1, \dots, n$, stop the algorithm, and let $\gamma_d^u \leftarrow \infty$ by lemma 2.
- (4) Reset $I=0$, and the number of iteration $z=1$.

Step 1: Tableau augmentation

- (1) If the supply a is increasing by Δs_a , add a dummy column, and allocate the flow Δs_a at the a^{th} row of the column. Let $c_{i,n+1} = 0, i = 1, 2, \dots, m$, and derive v_{n+1} , and the $\lambda_{i,n+1} = c_{ij} - u_i - v_{n+1}, i = 1, \dots, a-1, a+1, \dots, m$.
- (2) If the demand b is increasing by Δd_b , add a dummy row, and allocate the flow Δd_b at the b^{th} column of the row. Let $c_{m+1,j} = 0, j = 1, 2, \dots, n$, and derive the $\lambda_{m+1,j} = c_{ij} - u_{m+1} - v_j, j = 1, \dots, b-1, b+1, \dots, n$.
- (3) If the demand b is decreasing by Δd_b , let $x_{hb} \leftarrow x_{hb} - \Delta d_b$, where $h = \arg \max \{x_{ib} | x_{ib} > 0, i = 1, 2, \dots, m\}$. Then add a dummy column, and allocate the flow Δd_b at the h^{th} row of the dummy column. Let $c_{i,n+1} = 0, i = 1, 2, \dots, m$, and derive v_{n+1} , and the $\lambda_{i,n+1} = c_{ij} - u_i - v_{n+1}, i = 1, \dots, p-1, p+1, \dots, m$.
- (4) If the supply a is decreasing by Δs_a , let $x_{af} \leftarrow x_{af} - \Delta s_a$, where $f = \arg \max \{x_{aj} | x_{aj} > 0, j = 1, 2, \dots, n\}$. Then add a dummy row, and allocate the flow Δs_a at the f^{th} column of the dummy row. Let $c_{m+1,j} = 0, j = 1, 2, \dots, n$, and derive the $\lambda_{m+1,j} = c_{ij} - u_{m+1} - v_j, j = 1, \dots, q-1, q+1, \dots, n$.

Step 2: Labeling

- (1) Let the cell (k, l) as pivot cell, where $(k, l) = \arg \min \{\lambda_{ij} | \lambda_{ij} \leq 0\}$
 - (1.1) Let $\Gamma = \lambda_{kl}$. If number of iteration z is 1, then let $\Gamma = \Gamma'$.
 - (1.2) Proceed with the *labeling procedure* to obtain a close loop.
- (2) If there is no (k, l) to enter the basis, go to step 4.
 - (2.1) If the supply a is increasing by Δs_a , let $\Gamma = c_{a,n+1}$.
 - (2.2) If the demand b is increasing by Δd_b , let $\Gamma = c_{m+1,b}$.
 - (2.3) If the demand b is decreasing by Δd_b , let $\Gamma = c_{h,n+1} - c_{hb}$.
 - (2.4) If the supply a is decreasing by Δs_a , let $\Gamma = c_{m+1,f} - c_{af}$.

Step 3: Flow allocating

- (1) The close loop starts from the cell (k, l) .
- (2) Calculate $\theta = \min \{x_{ij} | x_{ij} \text{ is the even-number of the close loop except the value of } x_{ij} \text{ involving any } \Delta s_a \text{ or } \Delta d_b\}$
- (3) According to the close loop which is obtained by step 2, identify the chain reaction as follows:
$$x_{ij}^* = \begin{cases} x_{ij} + \theta, & \text{if } (i, j) \text{ is an odd-numbered of the close loop} \\ x_{ij} - \theta, & \text{if } (i, j) \text{ is an even-numbered of the close loop} \end{cases}$$
- (4) If a dummy column adds to the transportation tableau, $u_k \leftarrow -v_{n+1}$. Update the other u_i, v_j and λ_{ij} , however, keep v_{n+1} invariant.
- (5) If a dummy row adds to the transportation tableau, $v_l \leftarrow -u_{m+1}$. Update the other u_i, v_j and λ_{ij} , however, keep u_{m+1} invariant.

Step 4: Terminal condition checking

- (1) If $\Gamma' \neq \Gamma$, stop the algorithm, that the changing rate of objective function value is changed. Let $x_{ij} = 0$ and derive the magnitudes of Δs_a or Δd_b , then select the maximal value of Δs_a or Δd_b as the perturbation range of Δs_a or Δd_b .
- (2) If all λ_{ij} are greater than 0 in dummy row or in dummy column, and $\Gamma' = \Gamma$, stop the algorithm. Let γ_s^l as $-s_a; \gamma_d^l = -d_b$.
- (3) Otherwise, erase all the label and go to step 2, $z = z + 1$.

4. Numerical Example

The numerical example in this section, which has the multiple largest values of u_i and v_j , is a special case for demonstrating the effect of the proposed labeling algorithm. The optimal shipping pattern is shown in the Figure 1 that the optimal basic solutions are $x_{11} = 2, x_{12} = 3, x_{22} = 7, x_{23} = 5$, and $x_{31} = 8$.

4.1 The Perturbation Range for Increasing Supply 1

Iteration 1:

Step 0: Initialization

Reset $\Gamma = 0$. The number of iteration z is 1.

Step 1: Tableau augmentation

A dummy column with demand value Δs_1 is added in this tableau to balance the values of supply and demand because of supply 1 increasing.

| | | | | | |
|---|-----|----|-----|-----|----|
| | | 3 | 3 | 3 | |
| 0 | 3 | | 3 | 4 | 5 |
| | 2 | | 3 | (1) | |
| 1 | 5 | | 4 | 4 | 12 |
| | (1) | | 7 | 5 | |
| 1 | 4 | | 6 | 7 | 8 |
| | 8 | | (2) | (3) | |
| | 10 | 10 | 5 | | |

Figure 1: The primal and dual optimal solutions in a transportation tableau

Step 2: Labeling

In step 2, cell (2,4) is the pivot cell because of the negative reduced cost in the transportation tableau. By the labeling procedure, the close loop is determined which is (2, 4)-(2, 2)-(1, 2)-(1, 4) as shown in Figure 2 (a).

Step 3: Allocating the flow

Step 3 allocates the flow and determines the leaving variable. Because of $\theta = \min\{7\} = 7$, cell(2, 2) leaves the basis as shown in Figure 2 (b). $\Gamma' = \lambda_{24} = -1$ and let $\Gamma = \Gamma'$ in the first iteration. Let $u_2 = -v_4 = 0$, $v_3 = 4$, $\lambda_{21} = 2$, $\lambda_{22} = 1$, and $\lambda_{14} = 0$.

| | | | | | | |
|-----|---|-----|-----|-----|--------------|------------------|
| | | {1} | {1} | {2} | {2} | |
| | | 3 | 3 | 3 | 0 | |
| {4} | 0 | 3 | 3 | 4 | 0 | $5 + \Delta s_1$ |
| | | 2 | 3 | (1) | Δs_1 | |
| {2} | 1 | 5 | 4 | 4 | 0 | 12 |
| | | (1) | 7 | 5 | (-1) | |
| {3} | 1 | 4 | 6 | 7 | 0 | 8 |
| | | 8 | (2) | (3) | (-1) | |
| | | 10 | 10 | 5 | Δs_1 | |

(a)

| | | | | | | |
|---|-----|----|-----|--------------|------------------|------------------|
| | | 3 | 3 | 4 | 0 | |
| 0 | 3 | | 3 | 4 | 0 | $5 + \Delta s_1$ |
| | 2 | | 10 | (0) | $\Delta s_1 - 7$ | |
| 0 | 5 | | 4 | 4 | 0 | 12 |
| | (2) | | (1) | 5 | 7 | |
| 1 | 4 | | 6 | 7 | 0 | 8 |
| | 8 | | (2) | (2) | (-1) | |
| | 10 | 10 | 5 | Δs_1 | | |

(b)

Figure 2: (a) The first a close loop for the perturbation range of γ_s^u (b) The tableau after the flow allocating

Step4: Terminal condition checking

This algorithm assures to process the algorithm at least one iteration. Go to step 2 and erase all labels, $z = 1 + 1 = 2$.

Iteration2:

Step 2: Labeling

In step 2, cell (3, 4) is the pivot cell because of the negative reduced cost in the transportation tableau. By the labeling procedure, the close loop is determined which is (3, 4)-(3, 1)-(1, 1)-(1, 4) as shown in Figure 3 (a).

Step 3: Allocating the flow

Step 3 allocates the flow and determines the leaving variable. Because of $\theta = \min\{8\} = 8$, cell(3, 1) leave the basis as shown in Figure 3 (b). $\Gamma' = \lambda_{34} = -1$. Let $u_3 = -v_4 = 0$, $\lambda_{31} = 1$, $\lambda_{32} = 3$, and $\lambda_{33} = 3$.

Step4: Terminal condition checking

Because the changing rate of objective function value is invariant which is $\Gamma' = \Gamma = -1$, go to step 2, and erase all labels. $z = 2 + 1 = 3$.

Iteration3:

Step 2: Labeling

Since there is no candidate entering variable in Figure 3 (b), $\Gamma' = c_{14} = 0$, and go to step 4.

Step4: Terminal condition checking

The changing rate of objective function value Γ' is 0, which is different to $\Gamma = -1$, then stop the algorithm. Let $\Delta s_1 - 15 = 0$, which means $\Delta s_1 = 15$. The perturbation range of γ_s^u is 15.

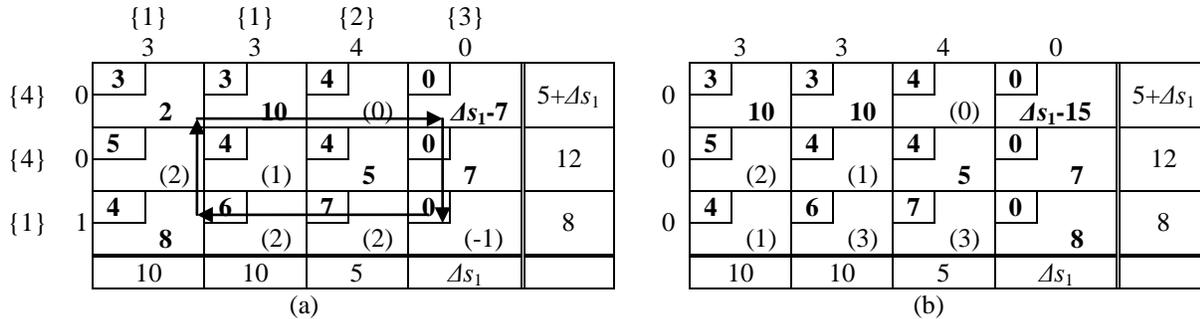


Figure 3: (a) The second close loop for the perturbation range of γ_s^u (b) The tableau after the flow allocating

4.2 The Perturbation Range for Decreasing Supply 2

Iteration1:

Step 0: Initialization

Reset $\Gamma = 0$. The number of iteration z is 1. Let $\Delta s_2 \geq 0$.

Step 1: Tableau augmentation

Because of supply 2 decreasing, a dummy row with supply value Δs_2 is added in this tableau to balance the values of supply and demand. Let the changing rate of objective function value $\Gamma = 0$. $h = 2$.

Step 2: Labeling

In step 2, cell(4, 1) is the pivot cell because of the zero reduced cost in the transportation tableau. By the labeling procedure, the close loop is determined which is (4, 1)-(4, 2)-(1, 2)-(1, 1) as shown in Figure 4.

Step 3: Allocating the flow

Step 3 allocates the flow and determines the leaving variable. Because of $\theta = \min\{2\} = 2$, let $\Delta s_2 \geq 2$, and cell(1, 2) leave the basis. $\Gamma' = \lambda_{41} = 0$, and let $\Gamma = \Gamma'$ in the first iteration. Since $v_1 = -u_4 = 3$, the reduced cost of this tableau is invariant.

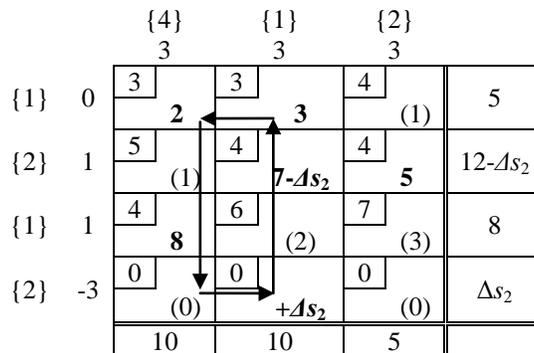


Figure 4: The first close loop for the perturbation range of γ_s^l

Step4: Terminal condition checking

This algorithm assures to process the algorithm at least one iteration. Go to step 2 and erase all labels, $z = 1 + 1 = 2$.

Iteration2:

Step 2 : Labeling

In step 2, cell (4, 3) is the pivot cell because of the zero reduced cost in the transportation tableau. By the labeling procedure, the close loop is determined which is (4, 3)-(4, 2)-(2, 2)-(2, 3) as shown in Figure 5(a).

Step 3: allocating the flow

Step 3 allocates the flow and determines the leaving variable. Because of $\theta = \min\{5\} = 5$, let $\Delta s_2 \geq 2 + 5 = 7$, and cell(2, 3) leave the basis as shown in Figure 5(b). $\Gamma' = \lambda_{43} = 0$. Since $v_3 = -u_4 = 3$, and the reduced cost of this tableau is invariant.

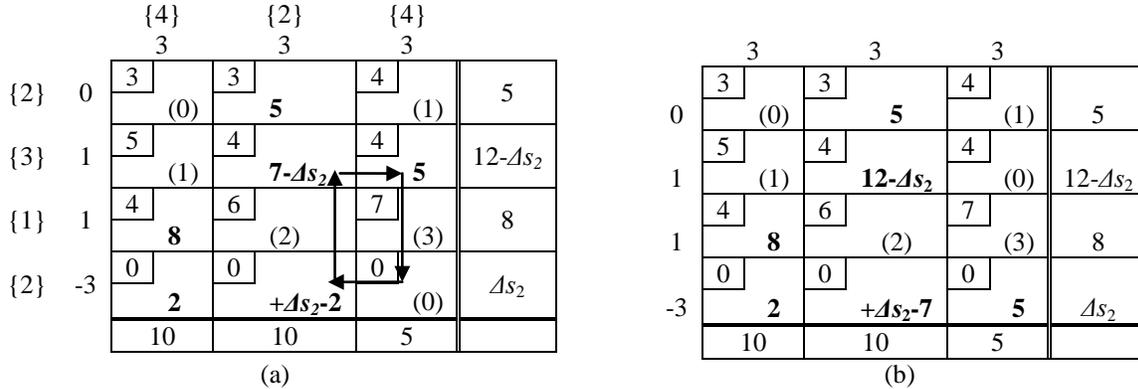


Figure 5: (a) The second close loop for the perturbation range of γ_s^l . (b) The tableau after the flow allocating

Step4: Terminal condition checking

Because the changing rate of objective function value is invariant which is $\Gamma' = \Gamma = 0$, go to step 2, and erase all labels. $z = 2 + 1 = 3$.

Iteration3:

Step 2: Labeling

Since there is no candidate entering variable in Figure 5 (b), $\Gamma' = c_{42} - c_{22} = 0 - 4 = -4$, and go to step 4.

Step4: Terminal condition checking

Because of no candidate entering variable, the algorithm is stop. Let $12 - \Delta s_2 = 0$ and $\Delta s_2 - 7 = 0$, which means $\Delta s_2 = 7$, and $\Delta s_2 = 12$. Therefore, select the maximal perturbation range, which γ_s^l is 12.

4.3 Discussion

All the perturbation ranges of Δs_i and Δd_j for the conventional basis invariant and the invariant changing rate of objective function value sensitivity analysis are summarized in the Table 2. With respect to Δs_1 , Δs_2 , Δd_1 , and Δd_2 , their lower bounds of the invariant changing rate of objective function value sensitivity analysis are lower than the basis invariant sensitivity analysis. Furthermore, with respect to Δs_1 , the upper bound of the invariant changing rate of objective function value sensitivity analysis is greater than the basis invariant sensitivity analysis. According to the result of invariant changing rate of objective function value sensitivity analysis, the upper bound of Δs_1 is not ∞ with respect to increase the RHS elements. The break point of the changing rate of Δs_1 is 15, which means the s_1 is the scarce resource for the transportation network. In the other hand, the increasing of the rest of RHS elements cannot improve the objective function value, which is regarded a kind of waste. In summary, the decision makers should spend more efforts on determining the values of scarce resources such as s_1 . If the flow is storable, then the flow in the dummy row or column might cause the holding cost, which is not considered in the TP. If the flow cannot be store, such as the electricity, the flow in dummy row or column perish immediately. Therefore, the rest of free available resources should set as minimum as possible.

Table 2: Perturbation ranges of RHS parameters for the basis invariant sensitivity analysis and the invariant changing rate of objective function value sensitivity analysis

| Type of sensitivity analysis | Δs_1 | Δs_2 | Δs_3 | Δd_1 | Δd_2 | Δd_3 |
|---|--------------|-----------------|----------------|-----------------|-----------------|----------------|
| Basis invariant | $[-3, 8]$ | $[-7, \infty)$ | $[-8, \infty)$ | $[-8, \infty)$ | $[-7, \infty)$ | $[-5, \infty)$ |
| Invariant changing rate of objective function value | $[-5, 15]$ | $[-12, \infty)$ | $[-8, \infty)$ | $[-10, \infty)$ | $[-10, \infty)$ | $[-5, \infty)$ |

5. Conclusion

The RHS parameters of the TP sometimes are determined by managerial decisions, which according to the marginal value of each RHS parameter. Cooperates or organizations expect to allocate the resources economically. The invariant changing rate of objective function value sensitivity analysis helps decision makers to determine which supply or demand value need to increase or decrease with maximal marginal value. This paper proposes two lemmas and develops a labeling algorithm to obtain the perturbation range for changing rate invariant of objective function value. A numerical example is also given to demonstrate that sensitivity analysis can provide boarder perturbation ranges comparing with the convention basis invariant sensitivity analysis. This result is hold under the assumption of primal non-degenerate, and lack of *more for less paradox*. The sensitivity analysis of the fuzzy TP is an interesting topic for future researches.

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Reference

- Aarvik, O., Randolph, P., The Application of Linear Programming to the Determination of Transmission Fees in an Electrical Power Network, *Interfaces*, vol. 6, no.1, pp. 47-49, 1975.
- Adlakha, V. G., Arsham, H., Distribution-Routes Stability Analysis of the Transportation Problem, *Optimization*, vol. 43, no. 1, pp. 47-72, 1998.
- Adlakha, V., Kowalski, K., A Quick Sufficient Solution to the More-For-Less Paradox in the Transportation Problem, *Omega*, vol. 26, no. 4, pp. 541-547, 1998.
- Adlakha, V., Kowalski, K., A Note on the Procedure MFL for a More-For-Less Solution in Transportation Problems, *Omega*, vol. 28, no. 4, pp. 481-483, 2000.
- Adlakha, V., Kowalski, K., Vemuganti, R. R., Lev, B., More-For-Less Algorithm for Fixed-Charge Transportation Problem, *Omega*, vol. 35, no.1, pp. 116-127, 2007.
- Arsham, H., Construction of the Largest Sensitivity Region for General Linear programs, *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1435-1447, 2007.
- Balinski, M. L., Gomory, R. E., A Primal Method for the Assignment and Transportation Problems, *Management Science*, vol. 10, no. 3, pp. 578-593, 1967.
- Bazaraa, M. S., Jarvis, J. J., Sherali, H. D., *Linear Programming and Network Flows*, 4th Edition, Wiley-Interscience, New York, 2010.
- Badra, N. M., Sensitivity Analysis of Transportation Problems, *Journal of Applied Sciences Research*, vol. 3, no. 8, pp. 668-675, 2007.
- Deineko, V. G.,Klinz, B., Woeginger, G., Which matrices are immune against the transportation paradox?, *Discrete Applied Mathematics*, vol. 130, no. 3, pp. 495-501, 2003.
- Doustdargholi, S., Derakhshan Asl, D., Abasgholipour, V., Sensitivity Analysis of Righthand-Side Parameter in Transportation Problem, *Applied Mathematical Sciences*, vol. 3, no. 30, pp. 1501-1511, 2009.
- Hadigheh, A. G., Terlaky, T., Sensitivity Analysis in Linear Optimization: Invariant Support Set Intervals, *European Journal of Operational Research*, vol. 169, no. 3, pp. 1158-1175, 2006.
- Hadigheh, A. G., Mirnia, K., Terlaky, T., Active Constraint Set Invariancy Sensitivity Analysis in Linear Optimization, *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 303-315, 2007.
- Hiller, F. S., Lieberman, G. J., *Introduction to Operations Research*, 9th Edition, McGraw-Hill, New York, 2010.
- Koltai, T., Terlaky, T., The Difference Between the Managerial and Mathematical Interpretation of Sensitivity Analysis Results in Linear Programming, *International Journal of Production Economics*, vol. 65, no. 3, pp. 257-274, 2000.
- Lin, C. J., Computing Shadow Prices/Costs of Degenerate LP problems with Reduced Simplex Tables, *Expert Systems with Applications*, vol. 37, no. 8, pp. 5848-5855, 2010.
- Lin, C.J., Determining Type II Sensitivity Ranges of the Fractional Assignment Problem, *Operations Research Letters*, vol. 39, no. 1, pp. 67-73, 2011.
- Lin, C.J., Wen, U.P., Lin, P. Y., Advanced Sensitivity Analysis of the Fuzzy Assignment Problem, *Applied Soft Computing*, vol. 11, no. 8, 2011,
- Schrenk, S., Finke, G., Cung, V.-D., Two Classical Transportation Problems Revisited: Pure Constant Fixed Charges and the Paradox, *Mathematical and computer modelling*, vol. 54, no. 9-10, pp. 2306-2315, 2011.
- Wu, N., Coppins, R., *Linear Programming and Extensions*, McGraw-Hill, New York, 1981.