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A Unified Model for Optimizing Inventory Decisions in a Centralized or Decentralized Supply Chain Comprising a Vendor without/with Lot Streaming and a Buyer with a Fixed Ratio of Partial Backorders

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Abstract

First of all, Yang et al.'s model is generalized by including three types of inspection cost for the vendor and a fixed backorder cost for the buyer, and unifying both situations: either having lot streaming or not for the vendor. Secondly, the optimal solutions to the two-stage centralized and decentralized models are individually derived, using differential calculus or algebraic methods. Thirdly, expressions for sharing the coordination benefits based on Goyal's (1976) scheme are derived. Fourthly, optimal expressions for some well-known models are deduced. Fifthly, two numerical examples for illustrative purposes are presented. Finally, some concluding remarks are drawn.

Keywords

Inventory; Production; Complete/Partial backorders; Differential calculus; The complete/perfect squares method

1. Introduction

Increasing attention has been given to management of a two- or multi- stage supply chain in recent years. This is due to rising competition, shorter life cycles of products and quick changes in today's business environment. The crux of a successful supply chain is the application of just in time (JIT) multiple deliveries. Through JIT-delivery agreement, enhancing efficiency, productivity and quality can be achieved. Goyal and Gupta (1989), Goyal and Deshmukh (1992), Bhatnagar et al. (1993), Maloni and Benon (1997), Sarmah et al. (2006) and Ben-Daya et al. (2008) have provided excellent reviews of the state-of-the-art of the joint economic lot-sizing problem arising in a two- or multi- stage supply chain. Assume that there is an uninterrupted production run. In the case of lot streaming in stage 1 (the vendor), shipments can be made from a production batch even before the whole batch is finished. However, the vendor cannot accommodate lot streaming because of regulations, material handling equipment, or production restrictions (Silver et al. 1998, p. 657). Without lot streaming, no shipments can be made from a production batch until the whole batch is finished. Sucky (2005) discussed the integrated single-vendor single-buyer system, with and without lot streaming, in detail. In many real life situations, demand may not be completely backordered during the stockout period, especially in a sales environment. Montgomery et al. (1973) first attempted to solve, using an *ad hoc* solution procedure so-called 'non-singular transformation', the *non-convex* cost function (with a slightly different formulation from that originated in Fabrycky and Banks 1967, pp. 92-96) also by considering a fraction β of demand during the stockout period to be backordered while the remaining fraction $(1-\beta)$ is lost. Rosenberg (1979) reformulated Montgomery et al.'s problem in terms of a fictitious demand rate. An optimal solution is sought using 'decomposition by projection' in the stepwise manner specified. Recently, Leung (2008a) has solved Montgomery et al.'s (1973) model using the methods of complete squares and perfect squares (which are algebraic approaches) in a more direct and much easier manner than 'non-singular transformation' or 'decomposition by projection' method. In addition, Leung's methods yield the optimal expressions from which we can straightforwardly identify three analytic cases that he considers to be much easier than using differential calculus (as claimed by Sphicas 2006 on page 59). Cardenas-Barron (2007, Table 1) and Leung (2009b) mention some papers with regard to solving certain inventory models without derivatives. In this paper, an integrated two-stage production-inventory system without/with lot streaming for the vendor and with a fixed ratio of partial backorders for the buyer, in which JIT deliveries are incorporated, is solved using differential calculus or algebraic methods; whereby four optimal expressions of three decision variables and the objective function are derived. Two numerical examples have been solved to illustrate the solution procedure, and five special cases are chosen and deduced from the unified model under study. Finally, some concluding remarks are made.

2. Assumptions, symbols and designations

The integrated production-inventory model is developed under the following assumptions:

1. A single item is considered.
2. There are two stages.
3. Production and demand rates (with the former greater than the latter) are independent of production or order quantity, and are constant.
4. Unit cost is independent of quantity purchased, and an order quantity will not vary from one cycle to another.
5. Neither a wait-in-process unit, nor a defective-in-transit unit, is considered.
6. The vendor implements perfect inspection to guarantee that defective units are not delivered to the buyer. Three types of inspection suggested in Wee and Chung (2007) are executed.
7. Lot streaming for the vendor is either allowed or not.
8. Shortages are allowed for the buyer and partially backordered (while the rest are lost sales), and all backorders are made up in the beginning of the next order cycle.
9. The vendor and the buyer have complete information of each other.
10. The number of the vendor's shipments is a positive integer.
11. A replenishment is instantaneous and a lead time is constant.
12. JIT deliveries for a two-stage supply chain are guaranteed.
13. The planning horizon is infinite.

The following symbols are used in the expression of the joint total relevant cost per year.

D = the buyer's demand rate [units per year]
P = the vendor's production rate [units per year]
h_1 = the vendor's linear holding cost [$ per unit per year]
h_2 = the buyer's linear holding cost [$ per unit per year]
s_1 = the vendor's fixed setup cost [$ per production cycle]
s_2 = the buyer's fixed cost of placing an order of any size [$ per order cycle]
a = the vendor's fixed inspection cost [$ per production cycle]
b = the vendor's variable inspection cost [$ per shipment]
c = the vendor's unit inspection cost [$ per unit]
$\overline{\pi}$ = the buyer's linear backorder cost [$ per backorder per year]
π_b = the buyer's fixed backorder cost [$ per backorder]
π_l = the buyer's fixed unit lost-sale cost (which is greater than π_b when the cost of goodwill is included) [$ per unit lost]
π_0 = the buyer's unit profit [$ per unit lost]
$\pi_{10} = \pi_1 + \pi_0$ = the buyer's fixed unit lost-sale cost plus unit profit [$ per unit lost]
β = the buyer's fraction of the demand is backordered during the stockout period, while the remaining fraction $(1-\beta) \equiv \overline{\beta}$ is lost

For a centralized supply chain (or the integrated approach), we have

R = the buyer's total demand per order cycle (a decision variable with non-negative real values) [units per order cycle]
S = the buyer's shortages or total demand per order cycle during the stockout period (a decision variable with non-negative real values) [units per order cycle]
$Q = R - \overline{\beta}S$ = the buyer's order quantity [units per order cycle]
K = the number of the buyer's orders or vendor's shipments per production cycle (a decision variable with positive integral values)
$TC_{1(\chi)}(K,R,S \mid \beta)$ = the vendor's total relevant cost, without/with lot streaming indicated by $\chi = 0/1$, per year as a function of K, R and S with a parameter β [$ per year]
$TC_2(R,S \mid \beta)$ = the buyer's total inventory cost as a function of R and S with a parameter β [$ per year]
$JTC_{(\chi)}(K,R,S \mid \beta)$ = the joint total relevant cost, without/with lot streaming, as a function of K, R and S with a parameter β (the objective function) [$ per year]

For a decentralized supply chain (or the independent approach), we have

r = the buyer's total demand per order cycle (a decision variable with non-negative real values) [units per order cycle]

s = the buyer's shortages or total demand per order cycle during the stockout period (a decision variable with non-negative real values) [units per order cycle]

$q = r - \overline{\beta}s$ = the buyer's order quantity [units per order cycle]

$k_{(\chi)}$ = the number of the buyer's orders or vendor's shipments per production cycle, dependent on whether having lot streaming or not (a decision variable with positive integral values)

$TC_2(r,s \mid \beta)$ = the buyer's total inventory cost as a function of r and s with a parameter β [$ per year]

$TC_1(k_{(\chi)} \mid \beta, r, s)$ = the vendor's total relevant cost as a function of $k_{(\chi)}$ with parameters β, r and s [$ per year]

To simplify the presentation of the subsequent mathematical expressions, we designate

$$\chi = \begin{cases} 0 & \text{without lot streaming} \\ 1 & \text{with lot streaming} \end{cases} \text{ and } \overline{\chi} = 1 - \chi, \tag{1}$$

$$\varphi = \frac{D}{P} \text{ and } \overline{\varphi} = 1 - \varphi, \tag{2}$$

where the former represents the proportion of production that goes to meet demand and the latter reflects the proportion of production allocated to inventory;

$$U \equiv U(K) = \frac{a + s_1}{K} + b + s_2, \tag{3}$$

$$V \equiv V(\beta) = \beta\pi_b + \overline{\beta}(\pi_{10} - c), \tag{4}$$

$$X \equiv X_{(\chi)}(K) = \varphi h_1[\overline{\chi}K - \chi(K-2)] + h_1(K-1) + h_2, \tag{5}$$

$$Y \equiv Y_{(\chi)}(K \mid \beta) = \varphi\overline{\beta}h_1[\overline{\chi}K - \chi(K-2)] + 0.5\overline{\beta}h_1(K-1) + h_2, \tag{6}$$

and

$$Z \equiv Z_{(\chi)}(K \mid \beta) = \varphi\overline{\beta}^2 h_1[\overline{\chi}K - \chi(K-2)] + \beta\overline{\pi} + h_2. \tag{7}$$

3. Solution to a unified model of a vendor-buyer supply chain

The vendor's average inventory level without/with lot streaming, which is the time-weighted inventory per production cycle length, is denoted and given by

$$I_{1(\chi)} = \frac{R - \overline{\beta}S}{2}\left\{(K-1) + \frac{\varphi(R - \overline{\beta}S)}{R}[\overline{\chi}K - \chi(K-2)]\right\}, \tag{8}$$

whose derivation is given in the Appendix A.

The vendor's total relevant cost per year is given by

$$TC_{1(\chi)}(K,R,S \mid \beta) = \frac{Ds_1}{KR} + h_1 I_{1(\chi)} + \frac{D[a + bK + cK(R - \overline{\beta}S)]}{KR}, \tag{9}$$

where the sum of the first two terms is the vendor's total inventory cost per year and the third term is the vendor's inspection cost per year.

Substituting equation (8) in (9) yields

$$TC_{1(\chi)}(K,R,S \mid \beta) = \frac{Ds_1}{KR} + \frac{h_1(K-1)(R - \overline{\beta}S)}{2} + \frac{\varphi h_1[\overline{\chi}K - \chi(K-2)](R - \overline{\beta}S)^2}{2R} + \frac{Da}{KR} + \frac{D(b - \overline{\beta}cS)}{R} + Dc. \tag{10}$$

The buyer's total inventory cost per year is given by

$$TC_2(R,S \mid \beta) = \frac{Ds_2}{R} + \frac{h_2(R-S)^2}{2R} + \frac{\beta\overline{\pi}S^2}{2R} + \frac{D(\beta\pi_b + \overline{\beta}\pi_{10})S}{R}. \tag{11}$$

Equation (11) is the expression for the economic order quantity (EOQ) model with partial backorders (see, e.g., Montgomery et al. 1973). This model is feasible provided

$$\sqrt{2Ds_2h_2} > D(\beta\pi_b + \overline{\beta}\pi_{10}) \quad \text{or} \quad \frac{\pi_{10} - \sqrt{\frac{2s_2h_2}{D}}}{\pi_{10} - \pi_b} < \beta \le 1. \tag{12}$$

Otherwise, it reduces to the classical EOQ model. Note that both derivation of feasibility condition (12) for the buyer and model reduction are given in Leung (2008a, 2009a).

Consequently, summing equations (10) and (11) yields the joint total yearly relevant cost for the integrated production-inventory system comprising a vendor without/with lot streaming and a buyer with a fixed ratio of partial backorders given by

$$JTC_{(\chi)}(K,R,S \mid \beta) = \frac{1}{2R}\begin{pmatrix} \{\varphi\overline{\beta}^2 h_1[\overline{\chi}K - \chi(K-2)] + \beta\overline{\pi} + h_2\}S^2 \\ -2\{\varphi\overline{\beta}h_1[\overline{\chi}K - \chi(K-2)] + 0.5\overline{\beta}h_1(K-1) + h_2\}RS + 2D[\beta\pi_b + \overline{\beta}(\pi_{10} - c)]S \end{pmatrix}$$
$$+ \frac{R}{2}\{\varphi h_1[\overline{\chi}K - \chi(K-2)] + h_1(K-1) + h_2\} + \frac{D}{R}\left(\frac{a+s_1}{K} + b + s_2\right) + Dc.$$

(13)

Incorporating designations (3) to (7) in equation (13) obtains

$$JTC_{(\chi)} \equiv JTC_{(\chi)}(K,R,S \mid \beta) = \frac{Z}{2R}\left[S^2 - \frac{2(YR - DV)S}{Z}\right] + \frac{XR}{2} + \frac{DU}{R} + Dc.$$

(14)

For a fixed positive integral value of the decision variable *K*, partially differentiating equation (14) with respect to *R* and *S* and then setting the two partial derivatives $\frac{\partial JTC_{(\chi)}}{\partial R}$ and $\frac{\partial JTC_{(\chi)}}{\partial S}$ to be zero yield

$$R^{\circ}_{(\chi)}(K \mid \beta) = \sqrt{\frac{D(2UZ - DV^2)}{XZ - Y^2}}, \tag{15}$$

$$S^{\circ}_{(\chi)}(K \mid \beta) = \frac{YR^{\circ}_{(\chi)}(K \mid \beta) - DV}{Z}, \tag{16}$$

and

$$JTC^{\circ}_{(\chi)}(K \mid \beta) \equiv JTC[K, R^{\circ}_{(\chi)}(K \mid \beta), S^{\circ}_{(\chi)}(K \mid \beta) \mid \beta]$$
$$= XR^{\circ}_{(\chi)}(K \mid \beta) - YS^{\circ}_{(\chi)}(K \mid \beta) + Dc \tag{17}$$
$$= \frac{\sqrt{D(2UZ - DV^2)(XZ - Y^2)} + DVY}{Z} + Dc. \tag{18}$$

Further, for a fixed *K*, equation (14) is a convex function and hence has a unique minimum, provided

$$2UZ > DV^2. \tag{19}$$

If convexity condition (19) for the integrated system is satisfied, then $Z > 0$ (because $U > 0$) and $XZ > Y^2$ (because $R^{\circ}_{(\chi)}(K \mid \beta) > 0$). Note that derivation of equations (15) to (18) and condition (19) by differential calculus is given in the Appendix A. Alternative derivation of equations (15) to (18) by algebraic methods is also given in the Appendix A.

The integrated model given by equation (14) is feasible, provided

$$S^{\circ}_{(\chi)}(K \mid \beta) > 0 \ \text{ iff condition (19) holds and } 2UY^2 > DV^2X, \tag{20}$$

whose derivation is given in the Appendix A.

Notice that in a numerical study (see Example 1 or 2 in Section 6 and the Appendix B), conditions (19) and (20) for $K = 1, 2, 3, \cdots$ are automatically checked when computing total demand and shortages given by equations (15) and (16).

Finally, two remarks on equation (17) or (18) are as follows:

(a) No closed-form expression for determining the optimal integral value of *K* can be derived from equation (17) or (18). However, the joint total relevant cost per year can be minimized by choosing $K = K^*_{(\chi)}$ such that

$$JTC^{\circ}_{(\chi)}(K \mid \beta) < JTC^{\circ}_{(\chi)}(K-1 \mid \beta) \quad \text{and} \quad JTC^{\circ}_{(\chi)}(K \mid \beta) \le JTC^{\circ}_{(\chi)}(K+1 \mid \beta).$$

(21)

The *global* minimum of the objective function is denoted by $JTC^*_{(\chi)} = JTC^{\circ}_{(\chi)}(K^*_{(\chi)} \mid \beta)$.

(b) When $JTC^{\circ}_{(\chi)}(1 \mid \beta) < JTC^{\circ}_{(\chi)}(2 \mid \beta)$, we have $K^{*}_{(\chi)} = 1$. This is due to the fact that when $K = 0$, we have $U(0) = \infty$ from designation (3), and hence $JTC^{\circ}_{(\chi)}(0 \mid \beta) = \infty$.

The buyer's EQQ is given by

$$Q^{*}_{(\chi)} = R^{*}_{(\chi)} - \overline{\beta} S^{*}_{(\chi)}, \tag{22}$$

where $R^{*}_{(\chi)} \equiv R^{\circ}_{(\chi)}(K^{*}_{(\chi)} \mid \beta)$ and $S^{*}_{(\chi)} \equiv S^{\circ}_{(\chi)}(K^{*}_{(\chi)} \mid \beta)$, and the vendor's economic production quantity (EPQ) is $K^{*}_{(\chi)} Q^{*}_{(\chi)}$.

4. Expressions for Sharing the Coordination Benefits

Recall that the buyer's total demand and shortages per order cycle, and the associated vendor's integer multiplier in a decentralized supply chain are denoted by *r*, *s* and $k_{(\chi)}$, respectively. Then equation (11) can be written as

$$TC_2(r, s \mid \beta) = \frac{Ds_2}{r} + \frac{h_2 (r-s)^2}{2r} + \frac{\beta \overline{\pi} s^2}{2r} + \frac{D(\beta \pi_{\mathrm{b}} + \overline{\beta} \pi_{10}) s}{r}. \tag{23}$$

which, on applying the methods of complete squares and perfect squares, yields the optimal values for the buyer given by

$$r^{*} = \sqrt{\frac{2Ds_2(h_2 + \beta\overline{\pi}) - [D(\beta \pi_{\mathrm{b}} + \overline{\beta} \pi_{10})]^2}{\beta \overline{\pi} h_2}}, \tag{24}$$

$$s^{*} = \frac{h_2 r^{*} - D(\beta \pi_{\mathrm{b}} + \overline{\beta} \pi_{10})}{h_2 + \beta \overline{\pi}}, \tag{25}$$

$$q^{*} = r^{*} - \overline{\beta} s^{*}, \tag{26}$$

and the resulting minimum total relevant cost per year given by

$$TC_2(r^{*}, s^{*} \mid \beta) = h_2 (r^{*} - s^{*}). \tag{27}$$

Note that derivation of equations (23) to (27) can be found in Leung (2008a, 2009a).

Assume that the demand for the item with which the vendor is faced is a stream of q^{*} units of demand. Given this stream of demand, Rosenblatt and Lee (1985, p. 389) showed that the vendor's EPQ should be some integer multiple of q^{*}. As a result, equation (10) can be written as

$$TC_1(k_{(\chi)} \mid \beta, r^{*}, s^{*}) = k_{(\chi)} \left\{ \frac{h_1 (r^{*} - \overline{\beta} s^{*})[r^{*} + \varphi(\overline{\chi} - \chi)(r^{*} - \overline{\beta} s^{*})]}{2r^{*}} \right\} + \frac{1}{k_{(\chi)}} \left[\frac{D(a + s_1)}{r^{*}} \right] - \frac{h_1 (r^{*} - \overline{\beta} s^{*})}{2} + \frac{\chi \varphi h_1 (r^{*} - \overline{\beta} s^{*})^2 + D(b - \overline{\beta} c s^{*})}{r^{*}} + Dc$$

$$= k_{(\chi)} \left\{ \frac{h_1 q^{*} [r^{*} + \varphi q^{*} (\overline{\chi} - \chi)]}{2r^{*}} \right\} + \frac{1}{k_{(\chi)}} \left[\frac{D(a + s_1)}{r^{*}} \right] - \frac{h_1 q^{*}}{2} + \frac{\chi \varphi h_1 (q^{*})^2 + D(b - \overline{\beta} c s^{*})}{r^{*}} + Dc. \tag{28}$$

Hence, the total yearly relevant cost in stage 1 can be minimized by choosing $k_{(\chi)} = k^{*}_{(\chi)}$ such that

$$TC_1(k_{(\chi)} \mid \beta, r^{*}, s^{*}) < TC_1(k_{(\chi)} - 1 \mid \beta, r^{*}, s^{*}) \quad \text{and} \quad TC_1(k_{(\chi)} \mid \beta, r^{*}, s^{*}) \le TC_1(k_{(\chi)} + 1 \mid \beta, r^{*}, s^{*}),$$

which, on following the derivation given in the Appendix of Leung (2009b), yields a closed-form expression for determining the optimal integral value of $k_{(\chi)}$ given by

$$k^{*}_{(\chi)} = \left\lfloor \sqrt{\frac{2D(a + s_1)}{h_1 q^{*} [r^{*} + \varphi q^{*} (\overline{\chi} - \chi)]} + 0.25} + 0.5 \right\rfloor. \tag{29}$$

Adopting Goyal (1976)'s judicious scheme for allocating the coordination benefits, we obtain some explicit expressions as follows:

$$\text{Share}_{i(\chi)} = (\text{Total saving})_{(\chi)} \times \frac{TC^*_{i(\chi)}}{\sum_{i=1}^2 TC^*_{i(\chi)}} = (\sum_{i=1}^2 TC^*_{i(\chi)} - JTC^*_{(\chi)}) \times \frac{TC^*_{i(\chi)}}{\sum_{i=1}^2 TC^*_{i(\chi)}},$$

(30)

where $TC^*_{1(\chi)} \equiv TC_1(k^*_{(\chi)} \mid \beta, r^*, s^*)$, $TC^*_{2(0)} = TC^*_{2(1)} \equiv TC_2(r^*, s^* \mid \beta)$ and $JTC^*_{(\chi)} \equiv JTC^\circ_{(\chi)}(K^*_{(\chi)} \mid \beta)$. Hence, the total yearly relevant cost, after sharing the benefits, in stage $i\ (=1, 2)$ is denoted and given by

$$TC^\circ_{i(\chi)} = TC^*_{i(\chi)} - \text{Share}_{i(\chi)} = JTC^*_{(\chi)} \times \frac{TC^*_{i(\chi)}}{\sum_{i=1}^2 TC^*_{i(\chi)}}. \tag{31}$$

In addition, the percentages of cost reduction in each stage and the entire supply chain are the same because $\frac{TC^*_{i(\chi)} - TC^\circ_{i(\chi)}}{TC^*_{i(\chi)}} = \frac{\text{Share}_{i(\chi)}}{TC^*_{i(\chi)}} = \frac{(\text{Total saving})_{(\chi)}}{\sum_{i=1}^2 TC^*_{i(\chi)}}$, and total saving and $\sum_{i=1}^2 TC^*_{i(\chi)}$ are constants.

5. Special Cases of the Two-Stage Centralized and Decentralized Supply Chains

Special case (1):

For $\chi = 1$, setting $\pi_\text{b} = 0$ and $a = b = c = 0$ in equations (13) and (15), (16), (18) yields the integrated model expressed by equation (2) and the optimal solution procedure in Section 3 of Yang et al. (2006), respectively.

For $\chi = 0$, parallel results can readily be deduced.

Special case (2):

Setting $\beta = 1.0$ in equation (13) yields the integrated single-vendor single-buyer model with not only linear but also fixed backorder costs.

Setting $\beta = 1.0$ in equation (18) yields

$$JTC^\circ_{(\chi)}(K \mid 1.0) = \sqrt{2D} \cdot \sqrt{\{\tfrac{a+s_1}{K} + [b + s_2 - \tfrac{D\pi_\text{b}^2}{2(\overline{\pi}+h_2)}]\}\{h_1[1+\varphi(\overline{\chi}-\chi)]K + [\tfrac{\overline{\pi}h_2}{\overline{\pi}+h_2} - h_1(1-2\chi\varphi)]\}} + \tfrac{Dh_2\pi_\text{b}}{\overline{\pi}+h_2} + Dc$$

$$= \sqrt{2D} \cdot \sqrt{\begin{aligned} &h_1[1+\varphi(\overline{\chi}-\chi)][b+s_2 - \tfrac{D\pi_\text{b}^2}{2(\overline{\pi}+h_2)}]K + \tfrac{a+s_1}{K}[\tfrac{\overline{\pi}h_2}{\overline{\pi}+h_2} - h_1(1-2\chi\varphi)] \\ &+ [b+s_2 - \tfrac{D\pi_\text{b}^2}{2(\overline{\pi}+h_2)}][\tfrac{\overline{\pi}h_2}{\overline{\pi}+h_2} - h_1(1-2\chi\varphi)] + h_1(1+\varphi)(a+s_1)\end{aligned}} + \tfrac{Dh_2\pi_\text{b}}{\overline{\pi}+h_2} + Dc.$$

Clearly, to minimize $JTC^\circ_{(\chi)}(K \mid 1.0)$ is equivalent to minimize

$$\phi(K) \equiv h_1[1+\varphi(\overline{\chi}-\chi)][b+s_2 - \tfrac{D\pi_\text{b}^2}{2(\overline{\pi}+h_2)}]K + (a+s_1)[\tfrac{\overline{\pi}h_2}{\overline{\pi}+h_2} - h_1(1-2\chi\varphi)]\tfrac{1}{K}.$$

Hence, the joint total relevant cost per year can be minimized by choosing $K = K^*_{(\chi)}$ such that

$$\phi(K) < \phi(K-1) \text{ and } \phi(K) \le \phi(K+1).$$

A closed-form expression, as derived in the Appendix of Leung (2009b), for determining the optimal integral value of *K* is denoted and given by

$$K^*_{(\chi)} = \left\lfloor \sqrt{\frac{(a+s_1)[\frac{\overline{\pi}h_2}{\overline{\pi}+h_2} - h_1(1-2\chi\varphi)]}{h_1[1+\varphi(\overline{\chi}-\chi)][b+s_2 - \frac{D\pi_\text{b}^2}{2(\overline{\pi}+h_2)}]} + 0.25} + 0.5 \right\rfloor, \tag{32}$$

where $\lfloor x \rfloor$ is the largest integer ≤ *x*.

Three immediate remarks follow:

(1) What is the value of $K^*_{(\chi)}$ if $b + s_2 - \frac{D\pi_\text{b}^2}{2(\overline{\pi}+h_2)} \le 0$, no matter whether $\frac{h_2\overline{\pi}}{h_2+\overline{\pi}} - h_1(1-2\chi\varphi) > 0$ or $\frac{\overline{\pi}h_2}{\overline{\pi}+h_2} - h_1(1-2\chi\varphi) \le 0$ in expression (32)? The buyer's optimal solution is to allow no shortages to occur. As a result, the integrated model given by equation (13) becomes that considered in Special case (5). The reason is that $b + s_2 - \frac{D\pi_\text{b}^2}{2(\overline{\pi}+h_2)} \le 0$ implies $\sqrt{2Ds_2h_2} < D\pi_\text{b}$, where $\sqrt{2Ds_2h_2}$ is the minimum average annual cost in the classical EOQ model.

(2) Given that $b+s_2-\frac{D\pi_b^2}{2(\bar{\pi}+h_2)}>0$, what is the value of $K_{(\chi)}^*$ if $\frac{\bar{\pi}h_2}{\bar{\pi}+h_2}-h_1(1-2\chi\varphi)\le 0$ in expression (32)? In such a case, $K_{(\chi)}^*=1$ because $\phi(K)$ or equivalently $JTC_{(\chi)}^{\circ}(K\,|\,1.0)$ is a strictly increasing function of *K* provided $\frac{\bar{\pi}h_2}{\bar{\pi}+h_2}-h_1(1-2\chi\varphi)\le 0$. This monotonic property is shown as follows: For any two positive integral values K_1 and K_2 (with $K_1 < K_2$), we have

$$\phi(K_2)-\phi(K_1)=(K_2-K_1)\{h_1[1+\varphi(\bar{\chi}-\chi)][b+s_2-\frac{D\pi_b^2}{2(\bar{\pi}+h_2)}]-\frac{a+s_1}{K_1K_2}[\frac{\bar{\pi}h_2}{\bar{\pi}+h_2}-h_1(1-2\chi\varphi)]\}>0.$$

Special case (3):

For $\chi=1$, setting $\beta=1.0$ and $\pi_b=0$ in equation (13) yields the integrated model expressed by equation (3a) of Wee and Chung (2007), where the first term should be $\frac{bB^2}{2q}$ (not $\frac{(b+H_b)B^2}{2q}$, which is doubly penalized each backorder). Hence, setting $\chi=1$, $\beta=1.0$ and $\pi_b=0$ in equations (15), (16), (18) and (32) yields the corresponding optimal expressions of (12a), (12b), (13) and (14) of Wee and Chung (2007), where the symbol *b* is replaced by $(b-H_b)$.

For $\chi=0$, parallel results can readily be deduced.

Special case (4):

For $\chi=1$, setting $\beta=1.0$, $\pi_b=0$ and $a=b=c=0$ in equations (13) and (15), (16), (18), (32) yields the integrated model expressed by equation (4) and the corresponding optimal expressions of (9), (10), (11), (15) of Wu and Ouyang (2003), respectively.

For $\chi=0$, parallel results can readily be deduced.

Special case (5):

For $\chi=1$, setting $\beta=1.0$, $S=0$ and $a=b=c=0$ in equation (13) yields the integrated model expressed by equation (2) of Yang and Wee (2002). Hence, setting $\chi=1$, $\beta=1.0$, $\pi_b=0$, $\bar{\pi}=\infty$ and $a=b=c=0$ in equations (15), (18) and (32) yields the corresponding optimal expressions given by equations (5), (6) and (10) of Yang and Wee (2002).

For $\chi=0$, parallel results can readily be deduced.

Finally, expressions (23) to (31) are valid for Special cases (1) to (5).

6. Numerical Examples

Example 1 (A two-stage centralized/decentralized supply chain, comprising a vendor without/with lot streaming and with inspections, and a buyer with a fixed ratio of partial backorders)
Suppose that an item has characteristics as follows:
D = 1000 units per year, P = 3200 units per year,
s_1 = \$400 per setup, s_2 = \$25 per order,
h_1 = \$4 per unit per year, h_2 = \$5 per unit per year,
$\bar{\pi}=\$5$ per backorder per year, $\pi_b=\$0.2$ per backorder,
$\pi_{10}=\$1$ (let $\pi_1=\$0.3$ and $\pi_0=\$0.7$) per unit lost,
a = \$50 per production, b = \$5 per shipment, c = \$0.05 per unit, and β = 0.7.

Table 1 shows the optimal results of the integrated approach, obtained using designations (2) to (7), and equations (15) to (17), (21) and (22). Detailed calculations to reach Table 1 are given in the Appendix B. Thus, the vendor's EPQ is 477.66 units for $\chi=0$ (or 516.82 units for $\chi=1$), the buyer's EOQ and maximum backorders are 477.66 (or 258.41) units and 214.24 (or 104.91) units, and the *global* minimum joint total relevant cost is \$1964.16 (or \$1941.07) per year.

When the ordering decision is governed by the buyer, Table 2 shows the optimal results of the independent approach, obtained using equations (24) to (29). Detailed calculations to reach Table 2 are also given in the Appendix B.

Table 3 shows the results after sharing the coordination benefits, obtained using equations (30) and (31). Column 2 of Table 3 shows that the centralized replenishment policy increases the costs of the buyer, while decreases the cost of the vendor. According to Goyal's (1976) saving-sharing scheme, the increased cost of the buyer must be covered so as to motivate him/her to adopt the centralized replenishment policy, and the total yearly saving of $526.64 (or $112.26) is shared to assure equal yearly cost reduction of 21.14% (or 5.47%) through both stages or the entire chain. Either situation has significant cost reduction and hence coordination is implementable; especially when no lot streaming is allowed for the vendor.

Table 1: Results for the centralized model, without (or with) lot streaming

Stage	Integer multiplier	EPQ/EOQ (units per cycle)	Max backorders (units per cycle)	Yearly cost ($)
Vendor	1 (or 2)	477.66 (or 516.82)	–	1091.32 (or 1317.56)
Buyer	−	477.66 (or 258.41)	214.24 (or 104.91)	872.85 (or 623.51)
Entire supply chain	−	−	−	1964.17 (or 1941.07)

Table 2: Results for the decentralized model, without (or with) lot streaming

Stage	Integer multiplier	EPQ/EOQ (units per cycle)	Max backorders (units per cycle)	Yearly cost ($)
Vendor	4 (or 5)	440.92 (or 551.15)	–	1995.30 (or 1557.80)
Buyer	−	110.23 (or 110.23)	11.12 (or 11.12)	495.55 (or 495.55)
Entire supply chain	−	−	−	2490.85 (or 2053.35)

Table 3: Results after sharing the coordination benefit, without (or with) lot streaming

Stage	Yearly saving ($) or penalty (–$)	Share ($ per year)	$\boldsymbol{TC}^{\circ}_{i(0)}$ (or $\boldsymbol{TC}^{\circ}_{i(1)}$) ($ per year)	Yearly cost reduction (%)
Vendor (i = 1)	903.98 (or 240.24)	421.90 (or 85.18)	1573.40 (or 1472.62)	21.14 (or 5.47)
Buyer (i = 2)	−377.30 (or −127.96)	104.78 (or 27.10)	390.77 (or 468.45)	21.14 (or 5.47)
Entire supply chain	526.68 (or 112.28)	526.68 (or 112.28)	1964.17 (or 1941.07)	21.14 (or 5.47)

Example 2 (A two-stage centralized/decentralized supply chain, comprising a vendor with lot streaming but without inspections, and a buyer with complete backorders or not feasible to have any shortages)
Consider the example in Yang et al. (2006) where the parameter values are almost the same as in Example 1, except $\pi_{\text{b}} = \$0$, $a = b = c = \$0$, and $\beta = 1.0$ (the complete backordering case) or $\beta = 0.5$ (a partial backordering case).

Detailed calculations to reach Tables 4, 5, 7, 8, 10 and 11 are all given in the Appendix B. Tables 6 and 9 show the results after sharing the coordination benefits, obtained using equations (30) and (31). Cost reductions of 1.14% and 0.48% are not significant and hence coordination is not attractive for either special case.

Table 4: Special case (4) – Results for the centralized model, with lot streaming

Stage	Integer multiplier	EPQ/EOQ (units per cycle)	Max backorders (units per cycle)	Yearly cost ($)
Vendor	2	526.24	–	1286.35
Buyer	−	263.12	131.56	423.91
Entire supply chain	−	−	−	1710.26

Table 5: Special case (4) – Results for the decentralized model, with lot streaming

Stage	Integer multiplier	EPQ/EOQ (units per cycle)	Max backorders (units per cycle)	Yearly cost ($)
Vendor	4	565.68	–	1378.86
Buyer	−	141.42	71.21	351.05
Entire supply chain	−	−	−	1729.91

Table 6: Special case (4) – Results of after sharing the coordination benefit, with lot streaming

Stage	Yearly saving ($) or penalty (–$)	Share ($ per year)	$\boldsymbol{TC}^{\circ}_{i(1)}$ ($ per year)	Yearly cost reduction (%)

Vendor (i = 1)	92.51	15.66	1363.20	1.14
Buyer (i = 2)	−72.86	3.99	347.06	1.14
Entire supply chain	19.65	19.65	1710.26	1.14

Table 7: Special case (5) – Results for the centralized model, with lot streaming

Stage	Integer multiplier	EPQ/EOQ (units per cycle)	Max backorders (units per cycle)	Yearly cost ($)
Vendor	5	551.70	–	1400.86
Buyer	−	110.34	0.00	502.42
Entire supply chain	−	−	−	1903.28

Table 8: Special case (5) – Results for the decentralized model, with lot streaming

Stage	Integer multiplier	EPQ/EOQ (units per cycle)	Max backorders (units per cycle)	Yearly cost ($)
Vendor	5	500	–	1412.50
Buyer	−	100	0.00	500.00
Entire supply chain	−	−	−	1912.50

Table 9: Special case (5) – Results after sharing the coordination benefit, with lot streaming

Stage	Yearly saving ($) or penalty (–$)	Share ($ per year)	$\boldsymbol{TC}_{i(\mathbf{1})}^{\circ}$ ($ per year)	Yearly cost reduction (%)
Vendor (i = 1)	11.64	6.81	1405.69	0.48
Buyer (i = 2)	−2.42	2.41	497.59	0.48
Entire supply chain	9.22	9.22	1903.28	0.48

Table 10: Results for the centralized model, with lot streaming and β = 0.5

Stage	Integer multiplier	EPQ/EOQ (units per cycle)	Max backorders (units per cycle)	Yearly cost ($)
Vendor	1	422.26	–	838.75
Buyer	−	422.26	187.52	862.76
Entire supply chain	−	−	−	1701.51

Table 11: Comparison of yearly cost structures
between complete backorders, no shortages and partial backorders

Cost components	⊗Complete backorders exist and are appropriately taken into account	No shortages exist and are appropriately taken into account centralized (or decentralized)	No shortages exist but are erroneously treated as partial backorders
Vendor's setup cost	*760.11*	*725.03 (or 800)*	*655.99*
Vendor's holding cost	*526.24*	*675.83 (or 612.5)*	*182.76*
Vendor's total cost	1286.35	1400.86 (or 1412.50)	838.75
Buyer's ordering cost	*95.01*	*226.57 (or 250)*	*41.00*
Buyer's holding cost	*164.45*	*275.85 (or 250)*	*225.92*
Buyer's backorder cost	*164.45*	*– (or –)*	*288.32*
Buyer's lost sales cost	*0.00*	*– (or –)*	*307.52*
Buyer's total cost	423.91	502.42 (or 500⊕)	862.76⊕
Joint total cost	1710.26	1903.28 (or 1912.50⊕)	1701.51⊕

Legends:

⊗ Column 2 of Table 11 indicates that column 2 of Table 1 in Yang et al. (2006) gives the incorrect numerical answers (e.g. an obvious error is that the lost sales penalty cost is zero but not $200.60).

⊕ The total saving $\$210.99\,(=1912.50-1701.51)$ of adopting the integrated model with $\beta=0.5$ cannot cover the increase $\$362.76\,(=862.76-500)$ of buyer's total cost. Hence, no coordination exists. By the way, coordination exists and the optimal inventory policy can be obtained using the integrated model with partial backorders for $0.5<\beta\leq 1$, which is obtained using condition (12). For more details, see Analytic case (i) in Leung (2009a, p. 556).

7. Concluding remarks

To solve the integrated model given by equation (13), differentiation or the methods of complete squares and perfect squares proposed in Leung (2008a, b) can be used. The latter are algebraic methods which require no working knowledge of multi-variable calculus and is usually taught when introducing the notion of a general quadratic equation in Form 2 at high school. Hence, the methods can be understood by ordinary readers. On the other hand, to investigate the monotonic property of an optimal expression as in Leung (2009a) or to examine the behavior (such as convexity condition (19) in the current paper) of the total cost function as in Chung (2008), the only resort is differential calculus.

Following the solution approach of a two-stage supply chain without/with lot streaming allowed for the vendor and with a fixed ratio of partial backorders allowed for the buyer in Section 3, we can solve, also using differential calculus or algebraic methods, the integrated model of a three-stage (see, e.g., Banerjee and Kim 1995, Khouja 2003, Chung and Wee 2007, and Ben-Daya and Al-Nassar 2008) or a four-stage (see, e.g., Leung 2009b, 2010a, b) supply chain with lot streaming allowed for some upstream firms (such as suppliers, manufacturers and assemblers) and with a fixed ratio of partial backorders allowed for some downstream firms (i.e. retailers).

Appendix A

A.1. Derivation of expression (8)

Without lot streaming, the vendor's time-weighted inventory and the buyer's inventory are depicted in Fig. 1. Because the production time per cycle is $\frac{KQ}{P}$ and the order time per cycle is $\frac{R}{D}$, the area of the triangle and the total area of the rectangles are respectively given by

$$\tfrac{1}{2}\cdot KQ\cdot\tfrac{KQ}{P}=\tfrac{(KQ)^2}{2P} \quad\text{and}\quad \tfrac{R}{D}[(K-1)Q+(K-2)Q+\cdots+Q+0]=\tfrac{(K-1)KQR}{2D}\,.$$

As the number of setups per year is $\frac{D}{KR}$, the vendor's average inventory level is given by

$$I_{1(0)}=\tfrac{D}{KR}[\tfrac{(KQ)^2}{2P}+\tfrac{(K-1)KQR}{2D}]$$
$$=\tfrac{Q}{2}[(K-1)+\tfrac{\varphi Q}{R}K]\text{, using designation (2).}$$

Notice that when $Q=R$, $I_{1(0)}=\frac{Q}{2}[(K-1)+\varphi K]$, which is the result of Goyal (1988, p. 237).

With lot streaming, the vendor's time-weighted inventory and the buyer's inventory are depicted in Fig. 2, from which we observe that

$$\text{time length of AE}=\tfrac{Q}{P},\ \text{time length of BF}=\tfrac{R}{D},\ \text{time length of CG}=\tfrac{R}{D}-\tfrac{Q}{P},$$
$$\text{time length of CH}=\tfrac{2R}{D}-\tfrac{Q}{P},\text{ and time length of DI}=\tfrac{2R}{D}-\tfrac{2Q}{P}.$$

The vendor's average inventory level, originally derived by Yang et al. (2006), is repeated as follows:

$$I_{1(1)}=\frac{\text{Time-weighted inventory}}{\text{Vendor's cycle time}}$$
$$=\frac{\text{Sum of areas } A_1, A_2, A_3,\cdots,A_n}{\text{Vendor's cycle time}}$$
$$=\tfrac{D}{KR}\{\tfrac{Q}{2}(\tfrac{Q}{P})+\tfrac{Q}{2}(\tfrac{2R}{D}-\tfrac{Q}{P})+\tfrac{Q}{2}(\tfrac{4R}{D}-\tfrac{3Q}{P})+\cdots+\tfrac{Q}{2}[\tfrac{2(K-1)R}{D}-\tfrac{(2K-3)Q}{P}]\}$$
$$=\tfrac{DQ}{KR}\{\tfrac{R}{2D}[2+4+\cdots+2(K-1)]-\tfrac{Q}{2P}[3+5+\cdots+(2K-3)]\}$$
$$=\tfrac{DQ}{2R}[\tfrac{(K-1)R}{D}-\tfrac{(K-2)Q}{P}]\text{, which is equation (1) of Yang et al. (2006)}$$
$$=\tfrac{Q}{2}[(K-1)-\tfrac{\varphi Q}{R}(K-2)]\,.$$

Notice that when $Q=R$, $I_{1(1)}=\frac{Q}{2}[(K-1)-\varphi(K-2)]$, which is the result of Joglekar (1988, pp. 1397-8).

Unifying the expressions of $I_{1(0)}$ and $I_{1(1)}$ by an indicator variable χ, and setting $Q=R-\overline{\beta}S$ yield expression (8).

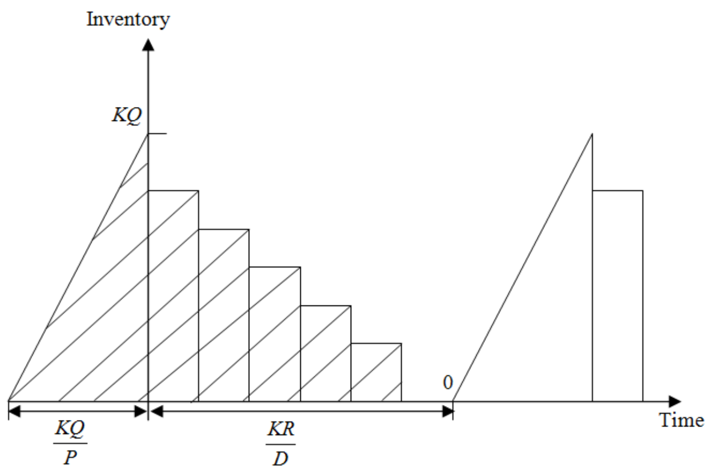

Fig 1. Without lot streaming, vendor's time-weighted inventory and buyer's inventory for K = 6.

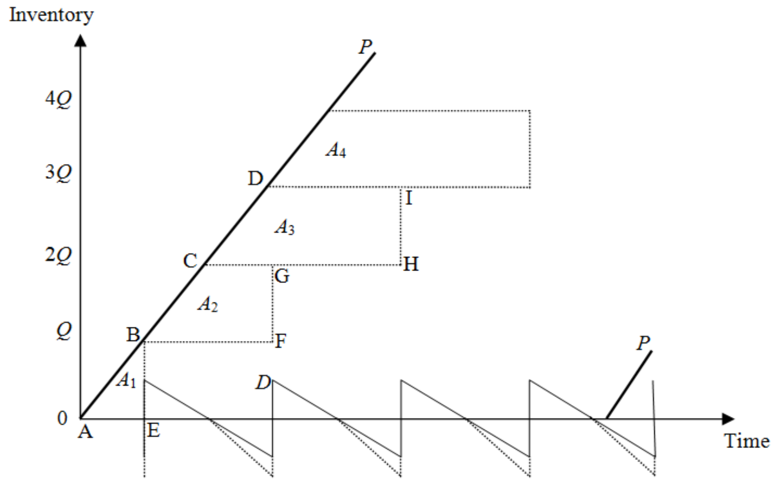

Fig 2. With lot streaming, vendor's time-weighted inventory and buyer's inventory for K = 4.

A.2. Derivation of equations (15) to (18) and condition (19) by differential calculus

Partially differentiating equation (14) with respect to R and S gives

$$\frac{\partial JTC_{(\chi)}}{\partial R}=\frac{-Z}{2R^2}[S^2-\frac{2(YR-DV)S}{Z}]-\frac{YS}{R}+\frac{X}{2}-\frac{DU}{R^2}\text{; hence } \frac{\partial JTC_{(\chi)}}{\partial R}=0 \text{ implies}$$

$$\frac{Z}{2R}[S^2-\frac{2(YR-DV)S}{Z}]+\frac{DU}{R}=\frac{XR}{2}-YS\,,$$

(33)

and

$$\frac{\partial JTC_{(\chi)}}{\partial S}=\frac{ZS}{R}-\frac{YR-DV}{R}\text{; hence } \frac{\partial JTC_{(\chi)}}{\partial S}=0 \text{ implies}$$

$$S=\frac{YR-DV}{Z}\,.$$

(16)

Substituting equation (16) in (33) yields

$$R=\sqrt{\frac{D(2UZ-DV^2)}{XZ-Y^2}}\,.$$

(15)

Substituting equation (33) in (14) obtains

$$JTC_{(\chi)}=XR-YS+Dc\,,$$

(17)

and then substituting equations (15) and (16) in (17) yields (18).

Partially differentiating the above two first-order derivatives with respect to R and S, and computing the determinant of the Hessian matrix H give

$$\frac{\partial^2 JTC_{(\chi)}}{\partial R^2}=\frac{ZS^2+2DVS+2DU}{R^3}\,,\ \frac{\partial^2 JTC_{(\chi)}}{\partial S^2}=\frac{Z}{R}\,,\ \frac{\partial^2 JTC_{(\chi)}}{\partial R\partial S}=-(\frac{ZS+DV}{R^2})\,,$$

and

$$|H| = \frac{\partial^2 JTC_{(\chi)}}{\partial R^2} \cdot \frac{\partial^2 JTC_{(\chi)}}{\partial S^2} - (\frac{\partial^2 JTC_{(\chi)}}{\partial R \partial S})^2 = \frac{D(2UZ - DV^2)}{R^4} > 0 \text{, provided}$$

$$2UZ > DV^2 .$$

(19)

A.3. Derivation of equations (15) to (18) by algebraic methods

Applying the complete squares method (by taking half the coefficient of *S*) to term 1 in equation (14) gives

$$JTC_{(\chi)} = \frac{Z}{2R}\left(S - \frac{YR - DV}{Z}\right)^2 + \frac{(XZ - Y^2)R}{2Z} + \frac{D(2UZ - DV^2)}{2ZR} + \frac{DVY}{Z} + Dc .$$

(34)

Temporarily assuming that $2UZ > DV^2$, $Z > 0$ and $XZ > Y^2$, and applying the perfect squares method to terms 2 and 3 in equation (34) obtain

$$JTC_{(\chi)} = \frac{Z}{2R}\left(S - \frac{YR - DV}{Z}\right)^2 + \left\{\sqrt{\frac{(XZ - Y^2)R}{2Z}} - \sqrt{\frac{D(2UZ - DV^2)}{2ZR}}\right\}^2 + \frac{\sqrt{D(2UZ - DV^2)(XZ - Y^2)} + DVY}{Z} + Dc.$$

(35)

For a fixed *K*, equation (35) has a unique minimum value when the two quadratic non-negative terms, depending on *R* or *S*, are made equal to zero. Therefore, the *locally* optimal values of these two decision variables and the *local* minimum value of the objective function are denoted and determined by equations (15), (16) and (18).

Equations (18) and (16) give $JTC_{(\chi)} = \frac{(XZ - Y^2)R}{Z} + \frac{DVY}{Z} + Dc$ and $\frac{DVY}{Z} = \frac{Y^2 R}{Z} - YS$. Substituting the latter expression in the former yields equation (17).

Equation (15) dictates either the case of $2UZ > DV^2$ and $XZ > Y^2$, or $2UZ < DV^2$ and $XZ < Y^2$. The former holds because cost model (14) must be given by a convex function, provided $2UZ > DV^2$ which implies $Z > 0$ and $XZ > Y^2$. This completes the algebraic derivation.

A.4. Derivation of condition (20)

If condition (19) holds, implying $Z > 0$ and $XZ > Y^2$; then

$$S^{\circ}_{(\chi)}(K \mid \beta) > 0 \Leftrightarrow YR^{\circ}_{(\chi)}(K \mid \beta) > DV \Leftrightarrow \frac{2UZ - DV^2}{XZ - Y^2} > \frac{DV^2}{Y^2} \Leftrightarrow 2UY^2 > DV^2 X .$$

Appendix B

B.1. Detailed calculations for Example 1

Because condition (12) is satisfied for $\beta = 0.7$, the buyer is feasible to have partial backorders.
Designations (2) to (4) give

$\varphi = \frac{1000}{3200} = 0.3125$; $U \equiv U(K) = \frac{450}{K} + 30$: $U(1) = 480$, $U(2) = 255$, $U(3) = 180$; $V \equiv V(0.7) = 0.425$.

Without lot streaming (i.e. $\chi = 0$) and $\beta = 0.7$, designations (5) to (7) give

$X \equiv X_{(0)}(K) = 5.25K + 1$: $X_{(0)}(1) = 6.25$, $X_{(0)}(2) = 11.5$, $X_{(0)}(3) = 16.75$;

$Y \equiv Y_{(0)}(K \mid 0.7) = 0.975K + 4.4$: $Y_{(0)}(1 \mid 0.7) = 5.375$, $Y_{(0)}(2 \mid 0.7) = 6.35$, $Y_{(0)}(3 \mid 0.7) = 7.325$;

$Z \equiv Z_{(0)}(K \mid 0.7) = 0.1125K + 8.5$: $Z_{(0)}(1 \mid 0.7) = 8.6125$, $Z_{(0)}(2 \mid 0.7) = 8.725$, $Z_{(0)}(3 \mid 0.7) = 8.8375$.

With lot streaming (i.e. $\chi = 1$) and $\beta = 0.7$, designations (5) to (7) give

$X \equiv X_{(1)}(K) = 2.75K + 3.5$: $X_{(1)}(1) = 6.25$, $X_{(1)}(2) = 9$, $X_{(1)}(3) = 11.75$;

$Y \equiv Y_{(1)}(K \mid 0.7) = 0.225K + 5.15$: $Y_{(1)}(1 \mid 0.7) = 5.375$, $Y_{(1)}(2 \mid 0.7) = 5.6$, $Y_{(1)}(3 \mid 0.7) = 5.825$;

$Z \equiv Z_{(1)}(K \mid 0.7) = 8.725 - 0.1125K$: $Z_{(1)}(1 \mid 0.7) = 8.6125$, $Z_{(1)}(2 \mid 0.7) = 8.5$, $Z_{(1)}(3 \mid 0.7) = 8.3875$.

Integrated approach (without lot streaming):

Equations (15) to (17) with $\chi = 0$, $K = 1$ or 2 and $\beta = 0.7$ give

$R^{\circ}_{(0)}(1 \mid 0.7) = \sqrt{\dfrac{1000[2\times 480\times 8.6125 - 1000(0.425)^2]}{6.25(8.6125)-(5.375)^2}} = \sqrt{\dfrac{8{,}087{,}375}{24.9375}} = 569.48$ units, where the positive value of the numerator in the square root sign indicates that convexity condition (19) is satisfied for $K = 1$;

$S^{\circ}_{(0)}(1 \mid 0.7) = \dfrac{5.375(569.48) - 1000(0.425)}{8.6125} = 306.06$ units, where the positive value of shortages indicates that feasibility condition (20) is satisfied for $K = 1$;

$JTC^{\circ}_{(0)}(1 \mid 0.7) = 6.25(569.48) - 5.375(306.06) + 1000(0.05) = \$1964.18$ per year.

$R^{\circ}_{(0)}(2 \mid 0.7) = \sqrt{\dfrac{1000[2\times 255\times 8.725 - 1000(0.425)^2]}{11.5(8.725)-(6.35)^2}} = \sqrt{\dfrac{4{,}269{,}125}{60.015}} = 266.71$ units, condition (19) is satisfied for $K = 2$;

$S^{\circ}_{(0)}(2 \mid 0.7) = \dfrac{6.35(266.71) - 1000(0.425)}{8.725} = 145.40$ units, condition (20) is satisfied for $K = 2$;

$JTC^{\circ}_{(0)}(2 \mid 0.7) = 11.5(266.71) - 6.35(145.40) + 1000(0.05) = \$2193.88$ per year.

Condition (21) dictates $K^{*}_{(0)} = 1$; thus $R^{*}_{(0)} = 569.48$ units, $S^{*}_{(0)} = 306.06$ units and hence maximum backorders $= 0.7(306.06) = 214.24$ units, and $JTC^{*}_{(0)} = \$1964.18$ per year.

Finally, equations (22), (10) and (11) give

$Q^{*}_{(0)} = 569.48 - 0.3(306.06) = 477.66$ units,

$$TC_{1(0)}(1, R^{*}_{(0)}, S^{*}_{(0)} \mid 0.7) = \frac{1000(400)}{569.48} + \frac{0.3125(4)(477.66)^2}{2(569.48)} + \frac{1000(50)}{569.48} + \frac{1000[5 - 0.3(0.05)(306.06)]}{569.48} + 50$$
$$= \$1091.32 \text{ per year,}$$

$$TC_2(R^{*}_{(0)}, S^{*}_{(0)} \mid 0.7) = \frac{1000(25)}{569.48} + \frac{5(263.42)^2}{2(569.48)} + \frac{0.7(5)(306.06)^2}{2(569.48)} + \frac{1000(0.44)(306.06)}{569.48} = \$872.85 \text{ per year,}$$

Total cost = 1091.32 + 872.85 = $1964.17 per year.

Independent approach by the buyer:
Equations (24) to (27) give

$r^{*} = \sqrt{\dfrac{2(1000)(25)(5+0.7\times 5) - [1000(0.7\times 0.2 + 0.3\times 1)]^2}{0.7(5)(5)}} = 114.99$ units,

$s^{*} = \dfrac{5(114.99) - 440}{8.5} = 15.88$ units; hence maximum backorders $= 0.7(15.88) = 11.12$ units,

$q^{*} = 114.99 - 0.3(15.88) = 110.23$ units,

$TC^{*}_{2(0)} = TC^{*}_{2(1)} = 5(114.99 - 15.88) = \$495.55$ per year.

Independent approach (without lot streaming):
Equations (29) and (28) give

$k^{*}_{(0)} = \left\lfloor \sqrt{\dfrac{2(1000)(450)}{4(110.23)[114.99 + 0.3125(110.23)]} + 0.25} + 0.5 \right\rfloor = \left\lfloor \sqrt{\dfrac{2(450{,}000)}{65{,}889.71} + 0.25} + 0.5 \right\rfloor = \lfloor 4.23 \rfloor = 4$,

$TC^{*}_{1(0)} = 4\left(\dfrac{65{,}889.71}{2\times 114.99}\right) + \dfrac{1}{4}\left(\dfrac{450{,}000}{114.99}\right) - \dfrac{4(110.23)}{2} + \dfrac{1000(5 - 0.3\times 0.05\times 15.88)}{114.99} + 50 = \$1995.30$ per year,

Total cost = 1995.30 + 495.55 = $2490.85 per year.

Integrated approach (with lot streaming):
Equations (15) to (17) with $\chi = 1$, $K = 1$ or 2 or 3 and $\beta = 0.7$ give

$R^{\circ}_{(1)}(1 \mid 0.7) = \sqrt{\dfrac{1000[2\times 480\times 8.6125 - 1000(0.425)^2]}{6.25(8.6125)-(5.375)^2}} = \sqrt{\dfrac{8{,}087{,}375}{24.9375}} = 569.48$ units, condition (19) is satisfied for $K = 1$;

$S^{\circ}_{(1)}(1 \mid 0.7) = \dfrac{5.375(569.48) - 1000(0.425)}{8.6125} = 306.06$ units, condition (20) is satisfied for $K = 1$;

$JTC^{\circ}_{(1)}(1\,|\,0.7) = 6.25(569.48) - 5.375(306.06) + 1000(0.05) = \$1964.18$ per year.

As expected, without or with lot streaming allowed for the vendor, the results are the same when $K = 1$.

$R^{\circ}_{(1)}(2\,|\,0.7) = \sqrt{\dfrac{1000[2\times 255\times 8.5 - 1000(0.425)^2]}{9(8.5)-(5.6)^2}} = \sqrt{\dfrac{4{,}154{,}375}{45.14}} = 303.37$ units, condition (19) is satisfied for $K = 2$;

$S^{\circ}_{(1)}(2\,|\,0.7) = \dfrac{5.6(303.37) - 1000(0.425)}{8.5} = 149.87$ units, condition (20) is satisfied for $K = 2$;

$JTC^{\circ}_{(1)}(2\,|\,0.7) = 9(303.37) - 5.6(149.87) + 1000(0.05) = \$1941.06$ per year.

$R^{\circ}_{(1)}(3\,|\,0.7) = \sqrt{\dfrac{1000[2\times 180\times 8.3875 - 1000(0.425)^2]}{11.75(8.3875)-(5.825)^2}} = \sqrt{\dfrac{2{,}838{,}875}{64.6225}} = 209.60$ units, condition (19) is satisfied for $K = 3$;

$S^{\circ}_{(1)}(3\,|\,0.7) = \dfrac{5.825(209.60) - 1000(0.425)}{8.3875} = 94.89$ units, condition (20) is satisfied for $K = 3$;

$JTC^{\circ}_{(1)}(3\,|\,0.7) = 11.75(209.60) - 5.825(94.89) + 1000(0.05) = \$1960.07$ per year.

Condition (21) dictates $K^*_{(1)} = 2$; thus $R^*_{(1)} = 303.37$ units, $S^*_{(1)} = 149.87$ units and hence maximum backorders $= 0.7(149.87) = 104.91$ units, and $JTC^*_{(1)} = \$1941.06$ per year.

Finally, equations (22), (10) and (11) give

$Q^*_{(1)} = 303.37 - 0.3(149.87) = 258.41$ units,

$TC_{1(1)}(2, R^*_{(1)}, S^*_{(1)}\,|\,0.7) = \dfrac{1000(400)}{2(303.37)} + \dfrac{4(258.41)}{2} + \dfrac{1000(50)}{2(303.37)} + \dfrac{1000[5 - 0.3(0.05)(149.87)]}{303.37} + 50 = \$1317.56$ per year,

$TC_2(R^*_{(1)}, S^*_{(1)}\,|\,0.7) = \dfrac{1000(25)}{303.37} + \dfrac{5(153.50)^2}{2(303.37)} + \dfrac{0.7(5)(149.87)^2}{2(303.37)} + \dfrac{1000(0.44)(149.87)}{303.37} = \$623.51$ per year,

Total cost = 1317.56 + 623.51 = \$1941.07 per year.

Independent approach (with lot streaming):

Equations (29) and (28) give

$$k^*_{(1)} = \left\lfloor \sqrt{\frac{2(1000)(450)}{4(110.23)[114.99 - 0.3125(110.23)]} + 0.25} + 0.5 \right\rfloor = \left\lfloor \sqrt{\frac{2(450{,}000)}{35{,}513.07} + 0.25} + 0.5 \right\rfloor = \lfloor 5.56 \rfloor = 5,$$

$$TC^*_{1(1)} = 5\left(\frac{35{,}513.07}{2\times 114.99}\right) + \frac{1}{5}\left(\frac{450{,}000}{114.99}\right) - \frac{4(110.23)}{2} + \frac{0.3125(4)(110.23)^2 + 1000(5 - 0.3\times 0.05\times 15.88)}{114.99} + 50$$

= \$1557.80 per year,

Total cost = 1557.80 + 495.55 = \$2053.35 per year.

B.2. Detailed calculations for Example 2

Because condition (12) is satisfied for β = 1.0, the buyer is feasible to have complete backorders. Thus, Special case (4) with $\chi = 1$ applies.

Special case (4) − Integrated approach (with lot streaming):

Setting $\chi = 1$, $\beta = 1.0$, $\pi_b = 0$ and $a = b = c = 0$ in equations (32), (15), (16), (18), (10) and (11) yields

$$K^*_{(1)} = \left\lfloor \sqrt{\frac{400[\frac{5\times 5}{5+5} - 4(1 - 2\times 0.3125)]}{4(0.6875)(25)} + 0.25} + 0.5 \right\rfloor = \lfloor 2.96 \rfloor = 2,$$

$R^*_{(1)} = Q^*_{(1)} = \sqrt{\dfrac{1000(2\times 225\times 10)}{9(10)-(5)^2}} = \sqrt{\dfrac{4{,}500{,}000}{65}} = 263.12$ units, condition (19) is satisfied for $K = 2$;

$S^*_{(1)} = \dfrac{5(263.12)}{10} = 131.56$ units, condition (20) is satisfied for $K = 2$;

$JTC^*_{(1)} = \dfrac{\sqrt{4{,}500{,}000(65)}}{10} = \$1710.26$ per year,

$$TC_{1(1)}(2, R^*_{(1)}, S^*_{(1)} \mid 1.0) = \frac{1000(400)}{2(263.12)} + \frac{4(263.12)}{2} = 760.11 + 526.24 = \$1286.35 \text{ per year,}$$

$$TC_2(R^*_{(1)}, S^*_{(1)} \mid 1.0) = \frac{1000(25)}{263.12} + \frac{5(131.56)^2}{2(263.12)} + \frac{5(131.56)^2}{2(263.12)} = 95.01 + 164.45 + 164.45 = \$423.91 \text{ per year,}$$

Total cost = 1286.35 + 423.91 = \$1710.26 per year.

Special case (4) – Independent approach (with lot streaming):
Setting $\chi = 1$, $\beta = 1.0$, $\pi_b = 0$ and $a = b = c = 0$ in equations (24), (25), (27), (29) and (28) yields

$$r^* = q^* = \sqrt{\frac{2(1000)(25)(5+5)}{5(5)}} = 141.42 \text{ units,}$$

$$s^* = \frac{5(141.42)}{5+5} = 71.21 \text{ units,}$$

$$TC^*_{2(1)} = 5(141.42 - 71.21) = \$351.05 \text{ per year,}$$

$$k^*_{(1)} = \left\lfloor \sqrt{\frac{2(1000)(400)}{4(141.42)[141.42 - 0.3125(141.42)]} + 0.25} + 0.5 \right\rfloor = \left\lfloor \sqrt{\frac{2(400{,}000)}{54{,}998.95} + 0.25} + 0.5 \right\rfloor = \lfloor 4.35 \rfloor = 4$$

$$TC^*_{1(1)} = 4\left(\frac{54{,}998.95}{2 \times 141.42}\right) + \frac{1}{4}\left(\frac{400{,}000}{141.42}\right) - \frac{4(141.42)}{2} + \frac{0.3125(4)(141.42)^2}{141.42} = \$1378.86 \text{ per year,}$$

Total cost = 1378.86 + 351.05 = \$1729.91 per year.

Because condition (12) is violated for β = 0.5, the buyer is not feasible to have any shortages. Thus, Special case (5) with $\chi = 1$ applies.

Special case (5) – Integrated approach (with lot streaming):
Setting $\chi = 1$, $\beta = 1.0$, $S = 0$, $\pi_b = 0$, $\bar{\pi} = \infty$ and $a = b = c = 0$ in equations (32), (15), (18), (10) and (11) yields

$$K^*_{(1)} = \left\lfloor \sqrt{\frac{400[5 - 4(1 - 2 \times 0.3125)]}{4(0.6825)(25)} + 0.25} + 0.5 \right\rfloor = \lfloor 5.04 \rfloor = 5,$$

$$R^*_{(1)} = Q^*_{(1)} = \sqrt{\frac{2(1000)(\frac{400}{5} + 25)}{17.25}} = \sqrt{\frac{210{,}000}{17.25}} = 110.34 \text{ units, } S^*_{(1)} = 0 \text{ units,}$$

$$JTC^*_{(1)} = \sqrt{210{,}000(17.25)} = \$1903.29 \text{ per year,}$$

$$TC_{1(1)}(5, Q^*_{(1)}) = \frac{1000(400)}{5(110.34)} + \left[\frac{4(4)(110.34)}{2} - \frac{0.3125(4)(3)(110.34)}{2}\right] = 725.03 + 675.83 = \$1400.86 \text{ per year,}$$

$$TC_2(Q^*_{(1)}) = \frac{1000(25)}{110.34} + \frac{5(110.34)}{2} = 226.57 + 275.85 = \$502.42 \text{ per year,}$$

Total cost = 1400.86 + 502.42 = \$1903.28 per year.

Special case (5) – Independent approach (with lot streaming):
Setting $\chi = 1$, $\beta = 1.0$, $s = 0$, $\pi_b = 0$, $\bar{\pi} = \infty$ and $a = b = c = 0$ in equations (24), (11), (29) and (10) yields

$$r^* = q^* = \sqrt{\frac{2(1000)(25)}{5}} = 100 \text{ units, } s^* = 0 \text{ units,}$$

$$TC^*_{2(1)} = \frac{100(25)}{100} + \frac{5(100)^2}{2(100)} = 250 + 250 = \$500 \text{ per year,}$$

$$k^*_{(1)} = \left\lfloor \sqrt{\frac{2(1000)(400)}{4(100)[100 - 0.3125(100)]} + 0.25} + 0.5 \right\rfloor = \left\lfloor \sqrt{\frac{2(400{,}000)}{27{,}500} + 0.25} + 0.5 \right\rfloor = \lfloor 5.92 \rfloor = 5$$

$$TC^*_{1(1)} = \frac{1000(400)}{5(100)} + \left[\frac{4(4)(100)}{2} - \frac{0.3125(4)(3)(100)^2}{2(100)}\right] = 800 + 612.5 = \$1412.50 \text{ per year}$$

Total cost = 1412.50 + 500 = \$1912.50 per year.

Violation of feasibility condition (12) for the buyer – Integrated approach (with lot streaming):

The fact that condition (12) is violated for $\beta = 0.5$ is disregarded. Designations (2) to (4) give $\varphi = \frac{1000}{3200} = 0.3125$; $U \equiv U(K) = \frac{400}{K} + 25$: $U(1) = 425$, $U(2) = 225$, $U(3) = \frac{475}{3}$; $V \equiv V(0.5) = 0.5$.
With lot streaming (i.e. $\chi = 1$) and $\beta = 0.5$, designations (5) to (7) give
$X \equiv X_{(1)}(K) = 2.75K + 3.5$: $X_{(1)}(1) = 6.25$, $X_{(1)}(2) = 9$, $X_{(1)}(3) = 11.75$;
$Y \equiv Y_{(1)}(K \mid 0.5) = 0.375K + 5.25$: $Y_{(1)}(1 \mid 0.5) = 5.625$, $Y_{(1)}(2 \mid 0.5) = 6$, $Y_{(1)}(3 \mid 0.5) = 6.375$;
$Z \equiv Z_{(1)}(K \mid 0.5) = 8.125 - 0.3125K$: $Z_{(1)}(1 \mid 0.5) = 7.8125$, $Z_{(1)}(2 \mid 0.5) = 7.5$, $Z_{(1)}(3 \mid 0.5) = 7.1875$.
Equations (15), (16) and (18) with $\chi = 0$, $K = 1$ or 2 and $\beta = 0.5$ give
$R^{\circ}_{(1)}(1 \mid 0.5) = \sqrt{\dfrac{1000[2 \times 425 \times 7.8125 - 1000(0.5)^2]}{6.25(7.8125) - (5.625)^2}} = \sqrt{\dfrac{6{,}390{,}625}{17.1875}} = 609.77$ units, condition (19) is satisfied for $K = 1$;
$S^{\circ}_{(1)}(1 \mid 0.5) = \dfrac{5.625(609.77) - 1000(0.5)}{7.8125} = 375.03$ units, condition (20) is satisfied for $K = 1$;
$JTC^{\circ}_{(1)}(1 \mid 0.5) = 6.25(609.77) - 5.625(375.03) = \$1701.52$ per year.
$R^{\circ}_{(1)}(2 \mid 0.5) = \sqrt{\dfrac{1000[2 \times 255 \times 7.5 - 1000(0.5)^2]}{9(7.5) - (6)^2}} = \sqrt{\dfrac{3{,}125{,}000}{31.5}} = 314.97$ units, condition (19) is satisfied for $K = 2$;
$S^{\circ}_{(1)}(2 \mid 0.5) = \dfrac{6(314.97) - 1000(0.5)}{7.5} = 185.31$ units, condition (20) is satisfied for $K = 2$;
$JTC^{\circ}_{(1)}(2 \mid 0.5) = 9(314.97) - 6(185.31) = \$1722.87$ per year.
Condition (21) dictates $K^*_{(1)} = 1$; thus $R^*_{(1)} = 609.77$ units, $S^*_{(1)} = 375.03$ units and hence maximum backorders $= 0.5(375.03) = 187.52$ units, and $JTC^*_{(1)} = \$1701.52$ per year.
Finally, equations (22), (10) and (11) give
$Q^*_{(1)} = 609.77 - 0.5(375.03) = 422.26$ units,
$TC_{1(1)}(1, R^*_{(1)}, S^*_{(1)} \mid 0.5) = \dfrac{1000(400)}{609.77} + \dfrac{0.3125(4)(422.26)^2}{2(609.77)} = 655.99 + 182.76 = \$838.75$ per year,

$$TC_2(R^*_{(1)}, S^*_{(1)} \mid 0.5) = \frac{1000(25)}{609.77} + \frac{5(234.74)^2}{2(609.77)} + \frac{0.5(5)(375.03)^2}{2(609.77)} + \frac{1000(0.5)(375.03)}{609.77}$$
$$= 41 + 225.92 + 288.32 + 307.52 = \$862.76 \text{ per year,}$$

Total cost = 838.75 + 862.76 = \$1701.51 per year.

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