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Robust Optimization Model for the Capacitated Facility Location and Transportation Network Design Problem

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Abstract

This paper investigates capacitated facility location-network design problem under uncertainty. A stochastic *p*-robust optimizing approach is applied to face the uncertainty. The developed model optimizes location for predefined number of capacitated facilities and underlying transportation network in such a that way total expected costs of links construction, facility construction, and transportation be minimized while the relative regret in each scenario must be no greater than an positive digit ($p \geq 0$). Uncertainty is characterized by given finite number of scenarios. Moreover, some basic behaviors of the problem are concluded through sensitive analysis from several random generated test problems.

Keywords
Stochastic *p*-robust optimization, capacitated facility location and network design problem, Sensitive analysis

1. Introduction

Facility location issues are managerial problem solving decisions with a wide range of applications in operational researches. The general objective of these problems is to locate one or more facilities that provide service to a set of demand points. The published papers in the location theory usually deal with locating facilities on a given network whereas, topology of transportation network has a considerable effect on transportation costs and optimal facility location and is often more cost-effective to change transportation network instead of adding extra facilities. Network design problems consider the decisions about network links construction and determining traffic flow on these links and probably satisfies additional constraints. Therefore, research's attention has been increased on developing the classical models to optimize facilities location and underlying transportation network at the same time. As a result, several models are proposed with diverse motivations. Main goal of such problems is to boost client's access to facilities by minimizing sum of the travel costs.

In this direction, Daskin et al. (1993) extended the uncapacitated fixed charge facility location problem to introduce Uncapacitated Facility Location/Network Design Problem (UFLNDP). Later on, Melkote and Daskin (2001) have developed this formulation to the Capacitated Facility Location-Network Design Problem (CFLNDP) which is much stronger than the previous formulation. Moreover, some heuristic methods have been proposed and an extensive sensitive analysis for this problem was presented. This is most closely related paper to our study with the absence of uncertainty. They also developed a model of maximum covering FLND problem with penalties for the uncovered demand (Melkote and Daskin (1998)). The CFLNDP has a large number of applications in many real-world applications such as: regional planning, supply chain and logistic problems, power transmission, pipeline distribution system, hub-and-spoke, and many other real-world problems.

Unfortunately, the CFLNDP has been considered with certain parameters while in lots decisions one is mostly encountered a set of unreliable or incomplete data which lead to make ineffective decisions. In the considered problem and also in other operational research applications, utilized data usually are noisy, erroneous, or carrying incomplete factors. Sensitivity analysis or stochastic programming models reveal defects of such data (Mulvey (1995)). Therefore, due to unpredictable future situation (e.g., change in market rate) and incomplete data, developing mathematical programming formulation to overcome these conditions is highly desirable.

Various approaches have been applied to overcome the uncertainty in real-problems [Snyder (2006), Baron et al. (2011)]. Classically, stochastic programming and robust optimization models are the most popular approaches for optimizing under uncertainty. Robust models are typically conservative and stochastic models include faults since decides are made on average. Furthermore, a new approach to deal with uncertainty which comes through the combination of the two above mentioned approaches is introduced by Snyder and Daskin [2006]. This approach is known as *stochastic p-robust* optimizing model. One of the main goals of this programming model is to buy more robust system with a little increase in costs. This approach minimizes the expected costs while robustness coefficient is incorporated into constraints. This approach is applied by the authors to overcome uncertainty.

Therefore, the considered problem optimizes the location for a predefined number of capacitated facilities and designing a transportation network which is subject to the robustness condition at the same time. The primarily goal of the authors is to minimize expected costs for opening facilities, construction links, and also travel costs. To be more realistic, most of the parameters of the model such as travel times or demands are assumed to be uncertain. So, it is believed that, the problem under consideration is more general and practical in numerous real-world logistic applications (Cocking and Reinelt (2006)).

2. Presenting the Model

In this section, the model formulation is illustrated. First, the following notations and assumptions are considered & defined and subsequently, the proposed model is presented.

Symbol	*Description*	*Indexed by*
N	set of potential nodes;	$i, j \in \{1, 2, ..., \|N\|\}$
L	Set of candidate links ;	$(i, j) \in L$
S	Set of scenarios;	$s \in \{1, 2, ..., \|S\|\}$
K	set of clients ;	$k \in \{1, 2, ..., \|N\|\}$
d_k^s	The demand of client k under scenario s ;	
u	Unit link construction cost;	
p	Robustness coefficient;	
f_i^s	Fixed cost of opening a facility on node i under scenario s **;**	
c_{ij}^s	Fixed cost of constructing a link (i, j) at scenario s	
t_{ij}^s	Travel cost per unit flow on link (i, j);	
$tr_{ij}^{ks} = d_k^s * t_{ij}^s$	Travel cost of client k on link (i, j) under scenario s;	
q_S	The probability that scenario s occurs;	
Z_s^*	Optimal solution for scenario s	
Ca_i	Capacity of each facility location	

Z_s^* is an input to the model that has been computed by solving each scenario separately (i.e., solves as deterministic CFLNDP problem) by using the CPLEX solver. The problem in hand includes various decisions, thus, following sets of decision variables are defined:

$$Z_i = \begin{cases} 1 & \text{if facility is open at node i;} \\ 0 & \text{Otherwise} \end{cases}$$

$$X_{ij} = \begin{cases} 1 & \text{if link } (i,j) \text{ is constructed;} \\ 0 & \text{Otherwise} \end{cases}$$

Y_{ij}^{ks} Fraction of commodity k traveling from i to j under scenario s;

W_i^{ks} Fraction of commodity k served by facility i under scenario s;

Note that here Z_i and X_{ij} unlike Y_{ij}^{ks} and W_i^{ks} are independent from index $\boldsymbol{s}$ to reflect two-stage nature of the problem. The following assumptions which are considered for the CFLNDP by Melkote, and Daskin [3] are also deemed at here. (1) each node represents a demand point, (2) facilities may be located only on the nodes of the network, (3) only one facility may be located per node, (4) the network is a customer-to-server system, in which the demands themselves travel to the facilities to be served, all network links are directed, (5) the links are uncapacitated (6) each node serves as both a customer and a potential facility site (i.e., $|N| = |K|$).

2.3 The proposed model

According to the mentioned assumptions, the mix-integer formulation of the studied problem is presented as follows.

$$Min \left\{ \sum_{s \in S} q_s * \left(\sum_{(i,j) \in L} \sum_{k \in N} tr_{ij}^{ks} * Y_{ij}^{ks} + \sum_{(i,j) \in L} X_{ij} * C_{ij}^{s} + \sum_{i \in N} f_i^{s} * Z_i \right) \right\} \quad \textbf{(1)}$$

Subject to:

$$Z_i + \sum_{j \in N} X_{ij} \geq 1; \qquad \forall i \in N \quad \textbf{(2)}$$

$$W_i^{is} + \sum_{j \in N} Y_{ij}^{is} = 1; \qquad \forall i \in N; \forall s \in S \quad \textbf{(3)}$$

$$\sum_{j \in N} Y_{ji}^{ks} = \sum_{j \in N} Y_{ij}^{ks} + W_i^{ks}; \qquad \forall i,k \in N; \forall s \in S : i \neq k \quad \textbf{(4)}$$

$$\sum_{i \in N} W_i^{ks} = 1; \qquad \forall k \in N, \forall s \in S \quad \textbf{(5)}$$

$$\sum_{k \in N} d_k^s * W_i^{ks} \leq K_i * Z_i; \qquad \forall i \in N, \forall s \in S \quad \textbf{(6)}$$

$$Y_{ij}^{ks} \leq X_{ij}; \qquad \forall k \in N, \forall s \in S, \forall (i,j) \in L \quad \textbf{(7)}$$

$$W_i^{ks} \leq Z_i; \qquad \forall i,k \in N; \forall s \in S \quad \textbf{(8)}$$

$$X_{ij} + X_{ji} \leq 1; \qquad \forall (i,j) \in L \quad \textbf{(9)}$$

$$\sum_{(i,j) \in L} \sum_{k \in N} tr_{ij}^{ks} * Y_{ij}^{ks} + \sum_{(i,j) \in L} X_{ij} * C_{ij}^{s} + \sum_{i \in N} f_i^{s} * Z_i \leq (1+p) * Z_s^*; \qquad \forall s \in S \quad \textbf{(10)}$$

$$\sum_{i \in N} Z_i = r; \quad \textbf{(11)}$$

$$Y_{ij}^{ks} \geq 0, \qquad X_{ij} \in \{0,1\}; \qquad \forall (i,j) \in L; \forall k \in N \ \& \forall s \in S \quad \textbf{(12)}$$

$$W_i^{ks} \geq 0, \qquad Z_i \in \{0,1\}; \qquad \forall i,k \in N : k \neq i \ \& \ \forall s \in S \tag{13}$$

Equation (1) refers to the objective function that minimizes total expected costs of travel cost, link construction cost, and facility construction cost, respectively. In constraint (2), each node can be serviced by multiple sources. That is, if the nearest facility or even its own facility could not satisfy it completely; single assignment property might be override and remaining fraction of demand should be transshipped to the other nodes by constructing extra links. In constraint (3) it is ensured by the authors that in each scenario, demand of node *i* is satisfied thoroughly, either by its own facility or by another facility located at other nodes. Therefore, sum of satisfied fraction of demand for client *i* in each scenario must be equal to one.

Constraint (4) is flow equilibrium which denotes income flow to the node *i* which is originated from node *k* ($i \neq k$) & must be equal in all scenarios to the amount of unsatisfied and satisfied demand for client *k* at this node. Constraint (5) states that total fraction demand which has been satisfied for client *k* through the network nodes, must be equal to one. On the other hand, each node must be completely satisfied in all scenarios. Since it is considered to have hard restriction on the facility's capacities and each located facility can only serve limited amount of demand, constraint (6) is to satisfy this issue which total demands served by facility located at node i, cannot exceed its capacity.

In addition, client *k* can establish flow on link (*i*,*j*) while it has been constructed before, that is represented by Constraint (7). Equation (8) shows that node *i* can serve demand for the client *k* if it was facility node. It worth noting that this constraint is useful for relaxation based methods to provide tighter bounds. Constraint (9) enforces links to be established only in one direction. As an example, flow cannot exist simultaneously from node *i* to node *j* and vice versa. Constraint (10) states the *p*-robustness condition. Relative regret in each scenario must be not greater than a constant value ($p \geq 0$). In other words, relative distance between cost of current solution and optimal cost of that scenario *s* must be within *p*. Therefore, in the proposed model the objective function of each scenario must be no greater than $(1+p)*Z_s^*$. Equation (11) states number of facilities which must be located. Finally, constraints (12-13) state type of decision variables as were explained in the previous section.

3. Sensitivity Analysis

In this section the sensitivity analysis is preformed to obtain main behavior of the problem. A random generated test problem with 20 nodes and 122 candidate links is utilized. Structure of the generated data is discussed in the following subsection.

3.1 Data generation

The locations of the clients were generated randomly and uniformly distributed over an $100*100$ area and potential connecting links between the clients were selected by random with a bias towards shorter links to emulate transportation network. In each data set, demands and fixed costs for scenario-1 were drawn uniformly from [0, 10000] and [4000, 8000] respectively and then rounded to the nearest integer. Link construction costs for scenario-1 were equal to Euclidean distances between facilities and customers multiplying to per-unit link construction cost ($u = 200$). Travel times are equal to the Euclidean distance between nodes at first scenario. Additional scenarios are identified by multiplying scenario-1 data to a random number drawn uniformly from [0. 5, 1.5] for all parameters. Also, it is assumed that link construction costs unlike travel costs, are symmetric. It worth noting that, five scenarios are considered (i.e. |S|=5) and *r*=0.1*|*N*|+1. For the sake of tractability, scenarios' weight (occurrence probability) are assumed to be equal (i.e., $q_s = 1/|S|; \forall s \in S$). Meanwhile, it is assumed that each facility has equal capacity (i.e. $Ca_i = Ca$) and minimum required capacity ($Ca_{\min}$) is calculated from Eq.(14).

$$Ca_{\min} = \frac{\max_{s \in S}\{\sum_{i \in N} d_i^s\}}{r} \tag{14}$$

3.2 number of facilities

Figure 1 demonstrates a trade off curve between number of facilities and expected costs. In each step the amount of capacity was equal to $1.05Ca_{\min}$. As the number of facilities is increasing the total expected costs and travel costs decreases until the number of facilities reaches to 14 facilities in which minimum expected costs is obtained. From 14 to 20 facilities, total expected costs and expected travel costs are increasing, the main reason of this behavior is due to decreasing in amount of facilities' capacity since the customers have to be assigned to the several facilities in order to be satisfied. Meanwhile, link construction cost and elapsed time due to high difficulty of the problem do not show a specific behavior.

3.3 Facilities' capacity

Facility capacities have a significant effect on different costs and also CPU run time. Therefore, the authors have increased the minimum capacity by 0.5% in each step. The obtained results are reported in Table 1. In this table *"%InCa"* denotes the percentage of increasing the minimum capacity. As expected, travel costs and link construction costs are decreasing since the larger amount of capacity customers can be served by the nearest facility and fewer extra links are required. It is noted that, the infinite amount of capacity problem becomes single-source problem instead of multiple-source one. Elapsed CPU run time decreased as well, since the single-source problem is easier than the multiple-source one.

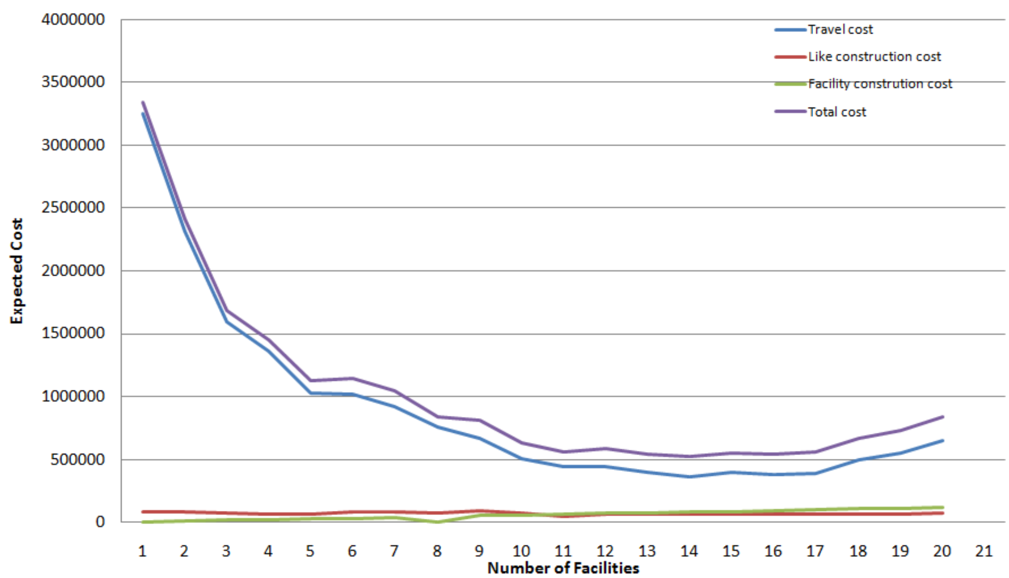

Figure 1: trade off curve between number of facilities and expected costs

Table 1: tradeoff between different costs and capacities

%In$_{Ca}$	*Travel cost*	*Link cost*	*Facility cost*	*Total cost*
0	1594536	73065	19599	1687200
0.5	1593368	73065	19599	1686031
1	1592201	73065	19599	1684865
1.5	1587425	76444	19599	1683468
2	1585705	75497	19599	1680801
2.5	1583479	72836	19599	1675914
3	1582898	72836	19599	1675333
3.5	1582317	72836	19599	1674752
4	1581737	72836	19599	1674172
4.5	1581194	70210	19599	1671003
5	1581149	66603	19599	1667351

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3.4 Expected costs *V.S* regret

One of the main goals of the stochastic *p*-robust programming models is to design a more robust system. In other words, large reduction in the maximum allowable regret makes only a little increase in the objective function (expected-cost). That is, more robust design can be bought with small increases in expected cost (Snyder and Daskin (2006)). A similar phenomenon for the problem in hand is observable. Therefore, the tradeoffs curve between the maximal relative regret (*p*) and the expected cost are similar to (Snyder and Daskin (2006)). Obtained results are summarized in Table 2 and the tradeoff curve is illustrated in Figure 2.

Table 2: Cost *vs.* Regret

p	*Obj value*	*%Inc*	*Max regret*	*%Dec*
∞	1667352	0	0.178	0.00
0.1777	1670639	0.20	0.158	10.91
0.1583	1675672	0.50	0.145	18.57
0.1447	1678260	0.65	0.120	32.41

The column labeled with the *"p"* gives the value of *p*, values under *"Obj value"* are the objective function, *"Inc%"* is to show the percentage of increase in the objective values, the *"Max regret"* column reports the maximum relative regret of the best solution found, and finally, *"Dec%"* indicates the percentage of decrease in the maximum regret. It is evident from the both Table 2 and Figure 2 that large reductions in the maximum regret are possible only with a little increase in the expected costs .As an example, 32.41 % reduction in maximum regret makes only 0.65 % increase in the expected costs.

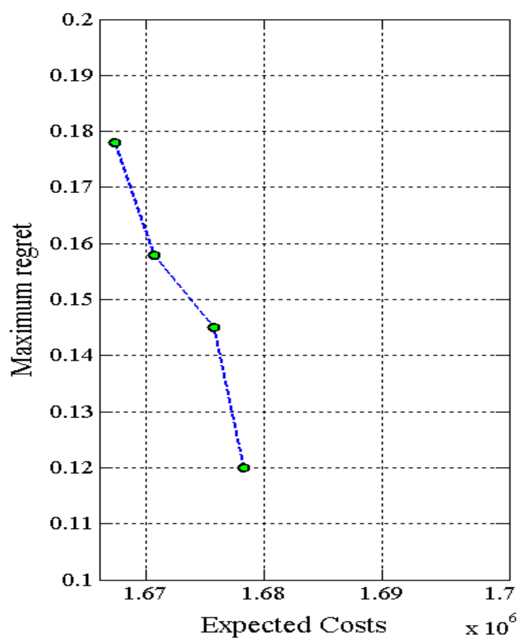

Figure 2: Tradeoff curve between the expected cost and maximum regret

4. Conclusion

In this study, an optimization model for the capacitated facility location/network design problem under uncertainty was presented. The presented model is a two-stage model in which at the first-stage decisions about facilities' location and underlying network are to be made and at the second-stage clients' assignment. One of the main goals of this programming model is to buy more robust system with a little increase in costs and the authors illustrated such phenomena for the proposed model. Several sensitivity analyses were performed in order to demonstrate the main behavior of the model by means of varying the key parameters such as number of facilities and amount of capacities. In this paper, it was assumed that links are uncapacitated but, at many real world applications (e.g., pipeline network), the network links have limitation on amount of flow they can carry. Hence, model can be generalized by considering capacitated links. Furthermore, the studied problem can be investigated as dynamic instead of static when the model parameters are varying during planning horizon.

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