

## **An Inventory Model with Demand Dependent Replenishment Rate for Damageable Item and Shortage**

**Mehdi Sajadifar and Ahmad Mavaji**  
**University of Science and Culture**  
**Poonak, Tehran, Iran**

### **Abstract**

Now a day, the inventory management for deteriorating items is one of the most important fields of production and sales researches. Inventory control for damageable/breakable items as a type of deteriorating items like glass, ceramic, etc. which get damaged during the storage due to accumulated stress of heaped stock can make a desired advantage for management in many industries. So these propositions put the retailer of breakable/damageable items in a conflicting situation between have large size and small size of order to minimize sum of cost of ordering, Holding, Shortage, and Breaking of stocks. There are some researches in field of Breakable items, but in all of them the shortage is not allowed. In this paper an inventory model for breakable/damageable item with allowing shortage is introduced which damaging rate is exponential and unsatisfied items will completely backlog. To solve this mathematical model a metaheuristic method, Simulated Annealing (SA), is applied.

### **Keywords**

Allow shortage, Breakable item, completely backlog, and simulated annealing technique.

### **Introduction**

Inventory problems for deteriorating items have been studied extensively by many researchers from time to time. The study in this area started with (1) which considered fashion goods deteriorating at the end of prescribed storage period. (2) Observed that items have been deteriorate in a negative exponential relation with time

$$\frac{d(I(t))}{d(t)} + \theta I(t) = -f(t)$$

Where  $\theta$  is the constant decay rate,  $I(t)$  the inventory level at the time  $t$ , and  $f(t)$  is the demand rate at time  $t$ . Deterioration refers to spoilage, dryness, vaporization, etc. of product (3).

But at the first time, (4) studied on a different type of deterioration called damaging/breaking. In the real life, there are some items, mainly made of glass, ceramics, china-clay, etc., which break/get damaged during the storage due to the accumulated stress of stocked items. So, deterioration is not a function of time, rather it has a dependency with inventory level at time. They introduced inventory model for breakable item with stock dependent demand in their model. (5) Considered inventory model for damageable item with stock dependent demand and variable replenishment rate. They considered problem in two scenarios where unit production cost is dependent with demand and breaking rate is stock level dependent in exponential form. (6) And (7) presented some inventory model for damageable item too. (8) considered inventory model for damageable item with stochastic demand.

For damageable items, there is a serious challenge for retailers. Dependency between damaging rate and inventory level, put the retailer/manufacturer of the items of glass, ceramic, china clay, etc. in a conflicting situation because he is tempted to go for a large stock to take advantage of less transport cost and decrease set up cost, but it lead to increase holding costs and invites more damage to his units, as damageability increases with the increase of piled stock. In the other hand, when shortage is allowed, it may increase average profit because of shortage costs advantage in face of holding inventory. According to these conditions, retailer/manufacturer has to try to make an optimal decision for maximum profit.

Inventory models for deterioration item with allowed shortage introduced by (9). (10) considered inventory model for deteriorating item with shortage. (11) Studied on perishable items with a stock-dependent demand in both scenario separates backlogging and lost sales. Also there are some papers with partial backlog concept. (12) Consider an inventory model whit constant demand for deteriorating item whit allowing shortage in partial backlog mode. But there is no research till now in inventory model for breakable/damageable items with allowed shortage.

The inventory control of damageable items is non-linear and complex especially because of nonlinear dependency between damaging and inventory level. So it is impossible to get the optimum solution by analytical solutions and researchers are forced to apply numerical optimization techniques for approximate optimum solution. There are some traditional methods for nonlinear problems but they face scientists to some problems like effort of initial solution, no efficiency for discrete variable and etc. metaheuristic algorithms particularly SA have been recently used as optimization techniques for non-linear complex problems.

Annealing is the physical process of heating up a solid until it melts followed by cooling it down slowly until it crystallizes into a state with a perfect lattice. Following this physical phenomenon, SA has been developed to find the global optimum for complex problems. In the early 1980s, (13) introduced the concepts of annealing in combinatorial optimization problems. Recently SA has been applied in different areas like TSP, Scheduling problems, Graph Coloring Problems, etc. for the first time (5) applied SA for inventory control for inventory control problem.

In this paper previous studies on damageable/breakable items developed with allowing the shortage when unsatisfied demand will be completely backlogged in each cycle; production rate is linearly dependent on demand. Also it is supposed that damaged goods have a price which is less than whole item obviously. We obtain optimum value of maximum inventory level and optimum level of shortage in each cycle to maximize profit function via simulated annealing method. The model is formulated in integral form with profit maximization principle concept. Here SA algorithm is used to find optimum inventory, shortage and scheduling period to maximize the profit. Model is illustrated with numerical values and sensitivity analysis due to the changes in the crucial parameters.

### Notation

Following notations are used for modeling the inventory problem:

|          |  |
|----------|--|
| Q        | maximum inventory level  |
| $Q_{10}$ | inventory level up to which no breaking occurs   |
| $Q_b$    | optimum level of allowed shortage (and it is positive).  |
| T        | duration length of each cycle = $t_1 + t_2 + t_3 + t_4 + t_5 + t_6$                                    |
| $T_1$    | duration length of production  |
| $T_2$    | duration length without production and without shortage  |
| $T_3$    | duration length when system face to shortage and have no production                                    |
| $T_4$    | duration length that begin when production restart and finishes when the inventory level reach to zero |
| Z        | average profit during time T   |
| $c_1$    | holding cost per unit quantity per unit time   |
| $c_s$    | shortage cost for each dissatisfied demand per unit time   |
| $c_3$    | set up cost per cycle  |
| $q(t)$   | inventory level at time t  |
| p        | production cost for each unit  |
| m        | mark-up of selling price for well item   |
| $m_0$    | mark-up of selling price for damaged item  |
| s        | selling price for well items   |
| $s_b$    | selling price for damaged items  |
| $B(q)$   | Number of damaged units per unit of time at time t and is a function of current inventory level q      |
| D        | demand rate per unit of time   |
| K        | replenishment / production rate = $b + cD$ where b, c being constant                                   |

### Assumptions

There are following assumptions to modeling this inventory problem

- 1) Demand is constant.
- 2) Breakable units  $B(q)$  is a function of current inventory level in polynomial form, i.e. (5)
 
$$B(q) = 0 \quad 0 \leq q \leq Q_{10}$$

$$= a(q)^\gamma \quad q \geq Q_{10}$$

Which  $a, \gamma (0 \leq \gamma \leq 1)$  are scale and shape parameters of breaking function.

- 3) Replenishment rate is linear function of demand

$$K = b + c \cdot D,$$

Where  $b > 0, c > 0$  are constant and value of  $K$  when system is in production time is always greater than sum of demand and breakage at time.

- 4) Time horizon is infinite.

- 5) Lead time is zero.

- 6) Selling price for whole number is a multiple of production cost:

$$s = m \cdot P \quad 0 < m < 1$$

- 7) Selling price for damaged items  $s_b$  is a multiple of production cost.

$$s_b = m_0 P \quad 0 < m_0 < 1$$

- 8) Shortage is allowed and demands which face to shortage will be backlogged. So total shortage cost is:  $c_s * \frac{Q_b * (\text{time length of shortage})}{2}$

The inventory cycle in general form is described as follow:

The initial stock at first of each cycle is zero and production start at the very beginning of cycle. Production rate is always greater than demand and breaking during system is producing. There is no breaking until the inventory level is less than  $Q_{10}$  (or until  $0 \leq t \leq t_1$ ). After that breaking is started and continued while inventory level reach down to  $Q_{10}$  again in rest of production period ( $t_3$ ). It means the inventory level increase because of production and decrease because of demand and shortage when system is in production time if inventory level be over than  $Q_{10}$  else decrease will only occur by demand. In the other hand, in the rest time of production, inventory level only decrease because of demand and damage if inventory level be over than  $Q_{10}$  else decrease will only occur by demand. In the rest time of production, shortage start when inventory level down to zero in  $t_4$ . In  $t_4 \leq t \leq t_5$  only demand affect on system and it is cumulated while inventory position reaches to  $Q_b$ . When production restart, system satisfy demand and piled shortage and finally satisfy all shortage and inventory level reach to zero (at time  $t_6$ ) and the cycle will be finished. Inventory cycle is shown in **Error! Reference source not found.**

### Mathematical formulation

There isn't any parametric solution for differential equations in breaking formulation. So here is used by some numerical methods for them. But wherever the equation could solve in parametric form, it has done.

The maximum inventory level  $Q$  may be state in one of these following situations:

Scenario 1:  $Q \leq Q_{10}$

Scenario 2:  $Q_{10} \leq Q$

In scenario 1 no damage will be occur and it converts to classic EPQ model. So, here it is not considered.

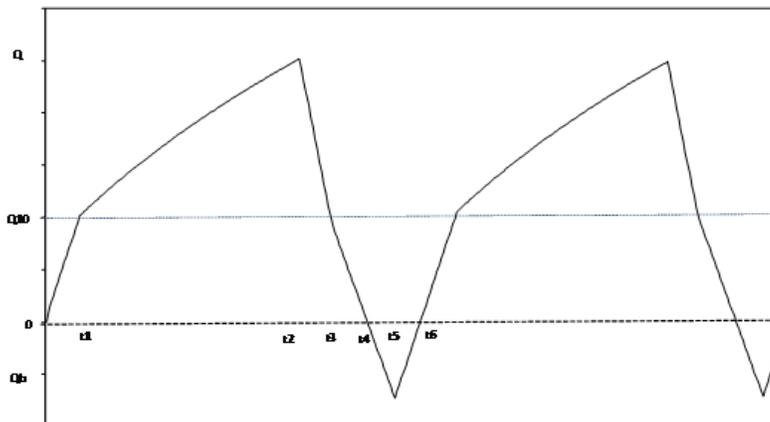


Fig 1- Inventory cycle

Scenario 2:

The differential equations to describe instantaneous inventory level where  $q(t)$  is the inventory level at time  $(t)$  and  $K > D + B$  is

$$\frac{dq}{dt} = \begin{cases} K - D & 0 < q < Q_{10} \\ K - D - B(q) & Q_{10} < q < Q \end{cases} \quad (1)$$

In production period and

$$\frac{dq}{dt} = \begin{cases} -D - B(q), & Q < q < Q_{10} \\ -D & 0 < q < Q_{10} \\ -D & Q_b < q < 0 \end{cases} \quad (2)$$

In rest time of production, including shortage time before restarting production. And

$$\frac{dq}{dt} = K - D \quad Q_b < q < 0 \quad (3)$$

From restart production to reach the inventory level to zero.

Time length of each cycle is  $T = T_1 + T_2 + T_3 + T_4$  where

$$T_1 = \int_0^{Q_{10}} \frac{dq}{K - D} + \int_{Q_{10}}^Q \frac{dq}{K - D - B(q)} \quad (4)$$

And

$$T_2 = \int_Q^{Q_{10}} \frac{dq}{-D - B(q)} + \int_{Q_{10}}^0 \frac{dq}{-D} \quad (5)$$

And

$$T_3 = \int_0^{-Q_b} \frac{dq}{-D(Q_{20})} \quad (6)$$

And

$$T_4 = \int_{-Q_b}^0 \frac{dq}{K - D} \quad (7)$$

Now, we should find benefits and costs relations in system.

Production cost: The production cost for each item is a function of demand at its production time. The total production cost formulate as below:

$$Pc = p * \left( \int_0^{Q_{10}} \frac{K \cdot dq}{K - D} + \int_{Q_{10}}^Q \frac{K \cdot dq}{K - D - B(q)} + \int_{-Q_b}^0 \frac{K \cdot dq}{K - D} \right) \quad (8)$$

Holding cost: The holding cost formulate as  $c_1 \cdot H(Q)$  where:

$$H(Q) = \int_0^{Q_{10}} \frac{q \cdot dq}{K - D} + \int_{Q_{10}}^Q \frac{q \cdot dq}{K - D - B(q)} + \int_Q^{Q_{10}} \frac{q \cdot dq}{-D(q) - B(q)} + \int_{Q_{10}}^0 \frac{q \cdot dq}{-D} \quad (9)$$

Shortage cost: Total shortage cost in each cycle is “shortage cost for each dissatisfied demand per unit time” multiple with “total shortage volume”

The total shortage volume is the area below inventory curve. So, value of triangle area which is below of zero line will be equal to total shortage value in each cycle. Therefore, total shortage cost is:

$$SHc = C_s * \left( \int_0^{-Q_b} \frac{q \cdot dq}{-D} + \int_{-Q_b}^0 \frac{q \cdot dq}{K - D} \right) \quad (10)$$

Selling price: total selling price in each cycle is formulated as:

$$s(Q) = m * p * \left( \int_0^{Q_{10}} \frac{D \cdot dq}{K - D} + \int_{Q_{10}}^Q \frac{D \cdot dq}{K - D - B(q)} + \int_Q^{Q_{10}} \frac{D \cdot dq}{-D(q) - B(q)} + \int_{Q_{10}}^0 \frac{D(q) \cdot dq}{-D(q)} + \int_{-Q_b}^0 \frac{K \cdot dq}{K - D} \right) \quad (11)$$

Breaking cost/sell: total number of damageable units is:

$$\theta(Q) = \int_0^T B(q) dt = \int_{Q_{10}}^Q \frac{B(q) \cdot dq}{K - D - B(q)} + \int_Q^{Q_{10}} \frac{B(q) \cdot dq}{-D(q) - B(q)} \quad (12)$$

Where  $B(q)$  is breaking rate when inventory level is  $q$ . and it can be substituted as follow

$$B(q) = \begin{cases} 0 & 0 < q < Q_{10} \\ aq^y & q > Q_{10} \end{cases}$$

As mentioned, selling price for each damaged item is:

$$s_b = m_0 * p \quad 0 < m_0 < 1$$

Which is a multiple of last production cost. So total selling price for damageable item is

$$\theta(Q) \cdot m_0 \cdot P(Q)$$

So finally, average profit is formulated as:

$$Z(Q) = \frac{[S(Q) + \theta(Q) \cdot m_0 \cdot P(Q) - Pc(Q) - c_3 - c_1 H_c(Q) - SHc]}{T} \quad (13)$$

#### Simulated Annealing procedure for model

- A. Representation and initialization** Tow real variable have been chosen for  $Q, Q_b$  as optimum inventory level (maximum inventory level) and optimum shortage level respectively. For initial solution two random real variables generate for  $Q, Q_b$  satisfying problem condition.

- B. Random transition presentation** A random number  $ran1$  between 1.00 and -1.00 is generated and  $Q + ran1$  is a neighborhood for  $Q$ . Again a random number  $ran2$  between 1.00 and -1.00 is generated and  $Q_b + ran2$  is a neighborhood for  $Q_b$ . If new value for  $Q, Q_b$  don't satisfy problem conditions, a new random neighborhood is generated.
- C. Energy function** The purpose is to find the optimum inventory level  $Q$  and the optimum shortage level as average profit  $Z(Q)$  is maximum. For correspondence with annealing algorithm,  $-Z(Q)$  is taken as the energy function.
- D. Cooling schedule** Initial temperature  $T_0$  is taken according to different parameter values of the energy function. Reducing factor  $C$  for  $T$  (temperature) is taken as 0.995.

**Numerical Results**

**Verification of algorithm.** If  $a = 0$  it means no damage has been occurred and it change to classic production model. For

$$D = 54.77, b = 60, c = 0.5, H = 2, c_3 = 200, c_s = 8, p = 13.61, m = 1.3$$

The exact optimum solution is  $Q = 63.9$  when shortage is not allowed and is  $Q = 57.1, Q_b = 14.2$  when shortage is allowed. Wherever with present SA algorithm it obtain  $Q = 63.9$  when shortage is not allowed and  $Q = 57.7, Q_b = 14.6$  when shortage is allowed.

**Solution of inventory model for scenario 2.** These values for problem parameters are used and results are shown in Table 1

$$D = 54.77, b = 60, c = 0.5, H = 2, c_3 = 200, c_s = 8, p = 13.61, m_0 = 0.15, m = 1.3, Q_{10} = 20$$

| Optimal solution solution for different value of $\gamma$ |         |       |       |         |       |
|---|---------|-------|-------|---------|-------|
| $\gamma$  | $T_1^*$ | $T^*$ | $Q^*$ | $Q_b^*$ | $Z^*$ |
| 0.2   | 1.4     | 3     | 42.3  | 18.0    | 88.3  |
| 0.25  | 1.3     | 2.8   | 39.2  | 17.1    | 88.2  |
| 0.3   | 1.2     | 2.8   | 36.9  | 18.4    | 81.9  |
| 0.4   | 1.0     | 2.4   | 30.3  | 18.3    | 75.6  |

**Table 1**

It is observed optimum value for  $Q^*$  decrease as  $\gamma$  increase, because as  $\gamma$  increase, number of damaged units increase and system want to have fewer inventories to have less damaged items. Also, with increasing damageability rate, obviously profit decreases. But rate of breakability hasn't a significant effect on number of shortage. Moreover, it is observed  $T_1^*$  decrease as  $\gamma$  increase because whit increasing the rate of damaging, system wants to have decrease time length of accumulating inventory.

**Sensitivity Analysis**

In this section, an analysis has been done in effect of fraction  $c_s/H$  on optimum solution and value of  $Q$  and  $Q_b$  in optimum situation. Hence, the plan for sensitive analysis is to increase shortage cost for each dissatisfied demand per unit time and comparing the result. For

$$D = 54.77, b = 60, c = 0.5, c_3 = 200, p = 13.61 \quad \gamma = 0.2$$

Results are given in the Table 2.

It is observed with increasing  $c_s/H$ , optimum value of average profit  $Z^*$  decrease. This occur because a parameter is increased which is cost type. Again it is interesting that  $Q^*$  increase and  $Q_b^*$  decrease when  $c_s/H$  become twofold. Since unit cost for backlog increase, system wants to decrease shortage cost via increase inventory on hand in constant holding cost, however just like to have some unsatisfied demand to gain more profit. When  $c_s/H$  is tripled, value of  $T$  become lower. It means when  $c_s/H$  is high, system tend to decrease duration between two cycle because of the tendency to have less number of backlogged item and as a result less shortage time.

| sensitivity analysis on $c_s/H$ |       |         |         |       |       |         |       |
|---------------------------------|-------|---------|---------|-------|-------|---------|-------|
| $H$                             | $c_s$ | $c_s/H$ | $T_1^*$ | $T^*$ | $Q^*$ | $Q_b^*$ | $Z^*$ |
| 2                               | 8     | 4       | 1.4     | 3.0   | 42.3  | 18.0    | 88.3  |

|   |    |   |      |     |      |      |      |
|---|----|---|------|-----|------|------|------|
| 2 | 12 | 6 | 1.62 | 3.0 | 47.7 | 13.1 | 81.5 |
| 2 | 16 | 8 | 1.60 | 2.9 | 48.3 | 9.3  | 78.1 |

**Table 2**

## Conclusion

This paper introduces an inventory model for damageable/breakable item demand dependant production rate and allowed shortage which unsatisfied demand is completely backlogged. Therefore, this paper present a practical solution for producer of breakable goods like ceramic, gypsum boards, glasses, etc. to make better decision at a situation when increase the number of inventory in hand cause more damaging. Moreover for the first time an inventory model for damageable item developed by allowing shortage in complete backlog mode.

This model can be extended to include partial backlog or completely lost sale. This problem can be also formulated in fuzzy. Moreover it can be developed by supposing delay in payment.

## References

- [1] Theory of Inventory Management. Whitin, T.M. 1957, Princeton University Press.
- [2] A model for exponentially decaying inventories. Ghare, P.N and Schrader, G.F. 1963, Journal of Industrial Engineering 15, pp. 238-243.
- [3] Recent trends in modeling of deteriorating inventory. Goyal, S. K. and Giri, B. C. 2001, European Journal of Operational Research.
- [4] Inventory of Damageable item with variable replenishment rate , stock-dependent demand and some unit in hand. Mandal, M and Maiti, M. 1999, Applied Mathematical Modeling 23, pp. 799-807.
- [5] Inventory of damageable items with variable replenishment and unit production cost via simulated annealing method. Maiti, M. k and Maiti, M. 2005, Computers and Industrial Engineering 49, pp. 432-448.
- [6] Inventory models for breakable items with stock dependent demand and imprecise constraints. Saha, A., et al. 2010, Mathematical and Computer Modelling 52, pp. 1771-1782.
- [7] Multi-item inventory model of breakable items with stock-dependent demand under stock and time dependent breakability rate. Guchhaiti, P, Maiti, K. M and Maiti, M. 2010, Computers and Industrial Engineering 59, pp. 911-920.
- [8] A stochastic discounted multi-objective solid transportation problem for breakable items using Analytical Hierarchy Process. Ojha, A, et al. 2010, Applied Mathematical Modelling 34, pp. 2256-2271.
- [9] A production–inventory model for a deteriorating item with trended demand and shortages. Sana, S, Goyal, S. K and Chaudhuri, K. S. 2004, European Journal of Operational Research 157, pp. 357-371.
- [10] A comparison between two pricing and lot-sizing models withpartial backlogging and deteriorated items. Teng, J. T, Ouyang, L. Y and Chen, L. 2007.
- [11] An EOQ model for items with stock dependent consumption rate and exponential decay. Padmanabhan, G and Vrat, P. s.l. : Engineering Costs and Production Economics, 1990, Engineering Costs and Production Economics 18, pp. 241-246.
- [12] A deterministic lot-size inventory model for deteriorating items with shortages and a declining market. Wee, H. M. 1995, Computers Operations Research 22, pp. 345-356.
- [13] Optimization by simulated annealing. Kirkpatrick, S, et al. s.l. : IBM Research Report, 1982.