Incorporating repeat purchasing in Innovation Diffusion Model Using Stochastic Differential Equations

Kuldeep Chaudhary[1], Kalyani Kumar[1] and Sunita Mehta[1, 2]
[1]Department of Applied Mathematics, Amity Institute of Applied Sciences
Amity University, Noida, 201301, India
[2]Department of Mathematics, Kalindi College
University of Delhi, India
kchaudhary26@amity.edu, kalyanikumar678@gmail.com, sunitasharmav78@gmail.com

Abstract

The existing marketing literature of Bass's innovation -diffusion model and many of its extended forms have been applied to depicting and predicting adoption curve for products. The loyalty towards a brand is often reflected when the products are bought by the consumer. In such cases, the theory of repeat purchasing exists and the higher is the repeat purchase value, it can be said that the better a firm is doing to keep customers loyal. In this paper, we use the concept of repeat purchase in an innovation and diffusion model based on $I\hat{t}o$ type of stochastic differential equation (SDE). Its applicability and accuracy is illustrated by means of new product sales data. Predictive validity and mean square error have been used to check the validity of the proposed model. It has been shown that proposed model performs comparatively better than SDE-based Bass innovation and diffusion model.

Keywords:
Innovation and diffusion model, Stochastic Differential equations, Repeat Purchasing

1. Introduction

In today's era of globalization, new products and technologies are being launched every day in the market by individuals, companies or organizations to keep up with the market and to save themselves from decline/fall. The major reasons why innovation is a survival skill are increasing competition, increased globalization, advanced technology, free trade, increased consumer expectations, changed workforce demographics, better communication channels and shorter life cycle of the products. However, at the same time, in the development and promotion of new product, risk of financial loss is also involved which calls for a judicious planning, risk analysis, market research and scientific decision–making.

Diffusion theory is a technique which describes how, why and at what rate a new product, idea or a service will be adopted in the society among consumers known as the potential adopter population. Since its introduction in 1960s, research work has been done by many of the researchers of various different fields to develop analytical models to predict the adoption of a new product amongst the potential consumers of the social system, Bass (1969) model being the earliest and most widely used for describing the diffusion of innovation. The diffusion of innovation is a process by which an innovation or an idea is communicated through certain channels over a period of time among the members of the social system. The main four diffusion variables in a diffusion process are firstly, the innovation itself, which is an idea, time, a practice or an object which is professed to be new by an individual, members of the social system or any other unit of adoption. The channels of communication are the means by which information is spread into or with in social system. Time is the rate at which innovation is diffused into the social system or is accepted by the members of the system. Individuals, companies and organizations which share a common culture and form a potential adopter population for innovation is known as the social system.

In the last several years, a considerable body of literature in the areas of innovation diffusion has been studied in depth by researchers from different disciplines, social sciences, economics and marketing (Easingwood, Mahajan and Muller, 1983; Mahajan and Peterson, 1985; Rogers, 1983). The main focus of the diffusion models in marketing is related to modelling and forecasting of the diffusion of innovations. The diffusion of a new product in marketing

is related to penetration of market. During the early stage of the diffusion process, some individuals alone are involved in adoption of the product through the external influence. We refer to these individuals as innovators and imitators are those who adopt the product through internal influences. Later, imitators adopt the product through word of mouth from adopters.
Since 1960s, a wide variety of pioneering work on diffusion processes has been studied in literature [Easingwood, Mahajan and Muller (1981, 1983), Floyd (1968), Lilien, Kotler and Moorthy(1998), Mahajan, Muller and Bass (1990), Mahajan and Peterson (1978) and Sharif and Ramanathan (1981)]. In general, they can be categorized into three categories viz. pure innovative, pure imitative and mixed influence. The Bass model (Bass, 1969) exhibits superior performance for the new product in predicting the sales growth phenomenon. To capture the variability in adoption behaviour of specific products, a number of modifications have been carried out in this model [Lilien, Kotler and Moorthy, 1998; Mahajan and Peterson, 1978, Easingwood, Mahajan and Muller, 1981; Floyd, 1968; Fisher and Pry (1971)]. Kapur et al (2004) proposed an alternative form of the Bass model by using a logistic time dependent rate function.

However, the literature on innovation-diffusion models in marketing has ignored stochastic considerations. The adoption process behaves as a stochastic process in continuous state space, if adoption process has smaller increments in total adoption as compared to the potential adopter population. Kapur et al (2012) proposed stochastic differential based Bass model incorporating the concept of change point.

The concept of repeat purchasing measures the loyalty of a consumer towards a particular brand as it is buying of a same product of the same brand name by a consumer which was bought by him earlier on a different occasion. Repeat purchasing is the most profitable way to increase revenue as it is easy to approach existing customers because of the connection that has already been made due to earlier purchases and also it costs more to attract new customers than to retain earlier customers. Therefore, a company or an organization needs to pay more attention towards the needs and demands of existing customers in order to increase repeat purchasing. In this paper, we extend the stochastic differential equation based Bass model using the concept of repeat purchasing in Innovation-Diffusion model.

The present paper is organized as follows. In section 2, we develop the framework of modeling of proposed model and solution of the proposed problem is obtained in closed form. Section 3 contains the parameter estimation results and goodness of fit curves. Section 4 concludes the paper.

2 Model Development

New products are being launched every day in market to meet the demands of consumers under the specified market conditions. The loyalty towards a brand is often reflected when the products are bought by the consumer. In such case, the theory of repeat purchasing exists and is an important real life phenomenon. Here, we develop a stochastic differential equation based innovation diffusion model for describing the diffusion of a new product due to both external and internal influences in market, considering repeat purchase. We assume that the size of the potential adopter population remains constant throughout of the product life cycle as determined in the beginning of the adoption process and repeat purchasing is influenced by both internal and external factors affecting first purchase. The successive increase in the number of adopters may consist of first purchase as well as repeat purchase of a product. If the product under considerations has a large life cycle, then as the time progresses, the adoption process has smaller increments in total adoption as compared to the potential adopter population. In such cases, adoption process defined by $\{N(t), t \geq 0\}$ behaves as a stochastic process with continuous state space. The rate of adoption of the product considering repeat purchase is described by the following differential :

$$\frac{dN(t)}{dt} = r(t)[\bar{N} - (1-g)N(t)] \quad (1)$$

Where $g(0 < g < 1)$ denotes proportion of the total adoption which is susceptible to repeat purchasing at any instant of time and $r(t)$ is a product adoption rate per remaining potential adopter at time t whose behaviour is not completely known since it is subject to random changes due to large number of factors such as promotional expenditure, product quality, change in customer preferences or competitors' strategies. Thus, $r(t) = b(t) + noise$ and equation (1) becomes:

$$\frac{dN(t)}{dt} = (b(t) + noise)[\bar{N} - (1-g)N(t)] \quad (2)$$

Where the term "noise" is denoted by "$\sigma\gamma(t)$", $\gamma(t)$ being the standard Gaussian white noise and $\sigma(>0)$ is the magnitude of the irregular fluctuations. Hence, equation (2) can be written as

$$\frac{dN(t)}{dt} = (b(t) + \sigma\gamma(t))[\bar{N} - (1-g)N(t)] \tag{3}$$

Using stochastic differential equation of an $I\hat{t}o$ type (Oksendal, 2003), equation (3) can be extended to the following

$$dN(t) = \left[b(t) - \frac{1}{2}\sigma^2\right][\bar{N} - (1-g)N(t)]dt + \sigma[\bar{N} - (1-g)N(t)]\, dW(t) \tag{4}$$

Where $W(t)$ is a one-dimensional Wiener process which is formally defined as an integration of the white noise $\gamma(t)$with respect to time t. Using $I\hat{t}o$ formula, solution to equation (4) with initial condition $N(0) = 0$, is given by:

$$N(t) = \frac{\bar{N}}{(1-g)}\left[1 - \exp\left\{(1-g)\int_0^t b(x)\ dx + \sigma\, W(t)\right\}\right] \tag{5}$$

In this proposed model, it is assumed that the product adoption rate $r(t)$ may change at any time moment and it can be defined as

$$r(t) = \begin{cases} \dfrac{b_1}{1+\beta_1 \exp(-b_1 t)} + \sigma\gamma(t) & for\ \ 0 \le t \le \tau \\ \dfrac{b_2}{1+\beta_2 \exp(-b_2 t)} + \sigma\gamma(t) & for\ \ t > \tau \end{cases} \tag{6}$$

Here we consider the case when the irregular fluctuations in adoption rate are same before and after the change-point. The corresponding stochastic differential equation for product adoption process can be written as

$$\frac{dN(t)}{dt} = \begin{cases} \left[\dfrac{b_1}{1+\beta_1 \exp(-b_1 t)} + \sigma\gamma(t)\right]\left[\bar{N} - (1-g)N(t)\right] & for\ \ 0 \le t \le \tau \\ \left[\dfrac{b_2}{1+\beta_2 \exp(-b_2 t)} + \sigma\gamma(t)\right]\left[\bar{N} - (1-g)N(t)\right] & for\ \ t > \tau \end{cases} \tag{7}$$

Therefore, the transition probability distribution of the above is obtained as follows.

$$N(t) = \begin{cases} \dfrac{\bar{N}}{(1-g)}\left[1 - \dfrac{(1+\beta_1)}{(1+\beta_1 \exp(-b_1 t))}\exp\left(-(1-g)[b_1 t + \sigma W(t)]\right)\right] & for\ \ 0 \le t \le \tau \\ \dfrac{\bar{N}}{(1-g)}\left[\begin{array}{l} 1 - \dfrac{(1+\beta_1)(1+\beta_2 \exp(-b_2 \tau))}{(1+\beta_1 \exp(-b_1 \tau))(1+\beta_2 \exp(-b_2 t))} \\ \exp\left(-(1-g)[b_1 \tau + b_2(t-\tau) + \sigma W(t)]\right) \end{array}\right] & for\ \ t > \tau \end{cases} \tag{8}$$

We consider the mean number adopters of product up to time *t*. As we know that the Brownian motion or Weiner Process follows normal distribution, thus the mean number of adopters is given as

$$E[N(t)] = \begin{cases} \dfrac{\bar{N}}{(1-g)}\left[1 - \dfrac{(1+\beta_1)}{(1+\beta_1 \exp(-b_1 t))}\exp\left((1-g)\left(-b_1 t + \dfrac{\sigma^2 t}{2}\right)\right)\right] & for\ \ 0 \le t \le \tau \\ \dfrac{\bar{N}}{(1-g)}\left[\begin{array}{l} 1 - \dfrac{(1+\beta_1)(1+\beta_2 \exp(-b_2 \tau))}{(1+\beta_1 \exp(-b_1 \tau))(1+\beta_2 \exp(-b_2 t))} \\ \exp\left((1-g)\left(-b_1 \tau - b_2(t-\tau) + \dfrac{\sigma^2 t}{2}\right)\right) \end{array}\right] & for\ \ t > \tau \end{cases} \tag{9}$$

3. Data Analysis and Model Validation

In order to test the validity and measure the performance of the proposed model, we have estimated the parameters of the proposed model on sales data of four products namely IBM Systems-in-use Generation-I (USA),sales data for air conditioners, telephone answering machine and colour television receivers cited in Kapur et al.(2012). Table1gives the description of all the four datasets with position of change point τ. The marketing strategy and promotional effort can be traced all the time during the life cycle of the product. The position of change point in adoption curve can be located by the analysis of adoption curve. Here, change points for the DS-I and DS-II has been fixed as eight and seventh year respectively. The proposed model chosen for comparative analysis is non-linear in nature so that method of non-linear regression applies for the estimation of parameters. We have estimated the parameters of the proposed model using SPSS tool based on nonlinear least square method. SPSS is statistical software package widely used for quantitative research. The applicability of the proposed model is determined by its ability of fitness on data used for analysis. For the comparison criteria, coefficient of multiple determination (R^2) and mean square error (MSE) has been used for goodness of fit of the proposed model.

3.1 Data description and estimation results

Table 1 gives the description of all four datasets. The values of estimated parameter of the proposed model incorporating repeat purchase given by equation (9) and without repeat purchase by Kapur et al (2012) for the all the data sets are given in Table2. The position of change point τ for all four data sets has been fixed as follow:

Data Set	Product	Position of change point
DS-I	IBM Systems-in-use Generation-I	$\tau = 8$
DS-II	Room Air Conditioner	$\tau = 7$
DS-III	Colour Television	$\tau = 6$
DS-IV	Telephone answering machine	$\tau = 8$

Table1: Data Description

DS-I		DS-II		DS-III		DS-IV	
1955	190	1	96	1	147	1	400
1956	750	2	291	2	585	2	895
1957	1750	3	529	3	1332	3	1474
1958	3430	4	909	4	2795	4	2171
1959	5972	5	1954	5	5441	5	3039
1960	8612	6	3184	6	10559	6	5133
1961	10962	7	4451	7	16336	7	7761
1962	12782	8	6279	8	22318	8	11067
1963	13952	9	7865	9	28280	9	15877
1964	14702	10	9538	10	32911	10	21432
1965	15157	11	11338				
1966	15460	12	12918				
1967	15663	13	14418				
1968	15833						
1969	15882						
1970	15911						
1971	15925						
1972	15931						
1973	15935						
1974	15939						
1975	15942						

Table2: Parameter Estimates and Comparison Criteria

Model under comparison	Parameter estimation	Data Set			
		DS-I	DS-II	DS-III	DS-IV
Kapur et al (2012)	$\bar{N}$	15933	18433	39787	54494
	b_1	.582565031	.232691	.580456	.024698
	b_2	.607863052	.371327	.631201	.459274
	β_1	23	14.8352	81.7412	1.04024
	β_2	27	30.6128	107.825	151.902
	σ	.132312	.113935	.111836	.000051
	R^2	.99974	.99934	.99970	.99969
	MSE	9154.32	10214.04	21021.02	7536.23
Proposed Model	$\bar{N}$	15933.023	22373.478	40000.000	84885.314
	b_1	.616	.376	.600	.312
	b_2	.559	.248	.627	.335
	β_1	26.225	36.724	81.741	60.520
	β_2	26.971	9.180	107.821	66.139
	σ	.199	.101	.112	.044
	g	.001	.013	.006	.009
	R^2	.99996	.99972	.99974	.99989
	MSE	1391.947	6704.002	43849.75	46322.9

3.2. Goodness of Fit Curves

The following figures show the goodness of fit of Bass model and the proposed model graphically.

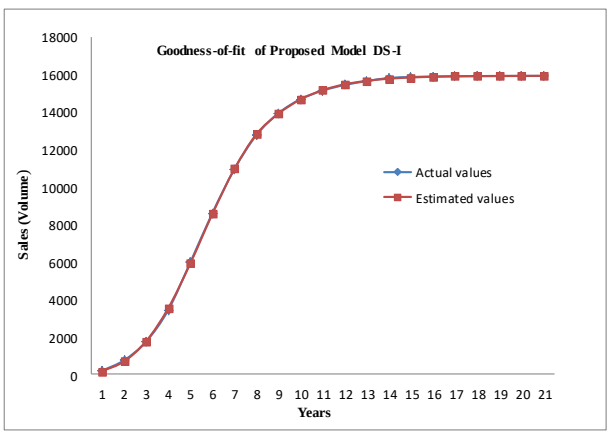

Fig 1: For DS-1

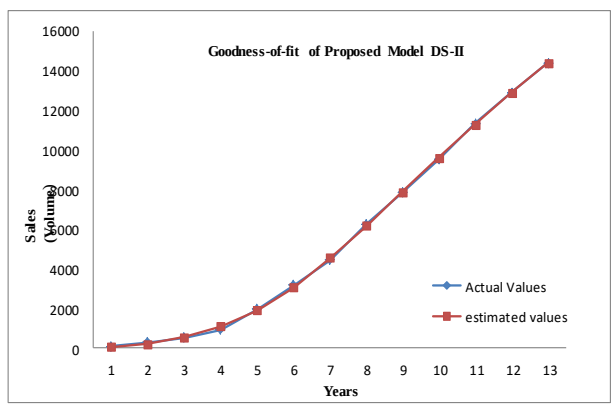

Fig 2: For DS-II

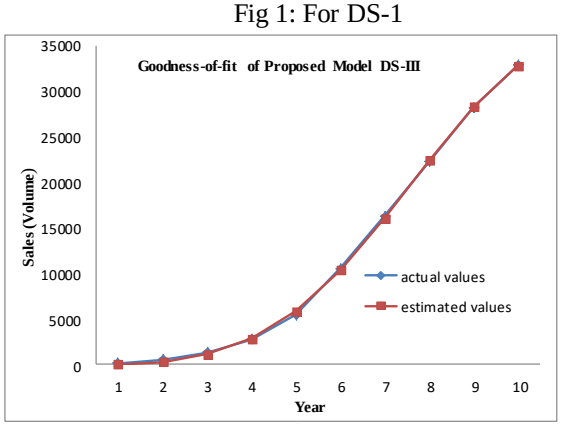

Fig3: For DS-III

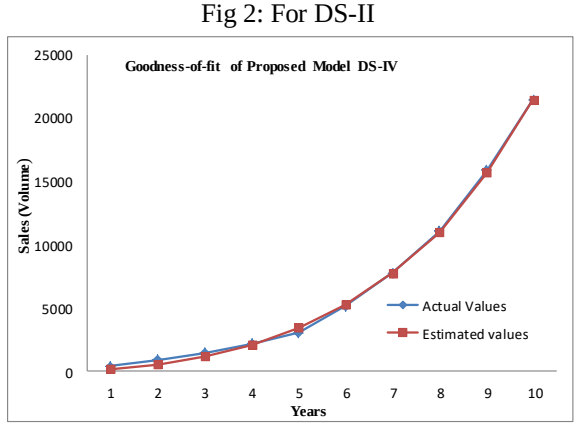

Fig4: For DS-IV

4. Discussion and Conclusion

In Table 2, we summarize the values of parameter estimation. From the Table 2, it can be seen that the proposed model describes the adoption growth and fits in all the data sets and performs better than Kapur et al (2012) stochastic differential based Bass model. If we discuss about the proposed model, we have seen that proposed model fits best in data set DS-I and DS-II with lower value of MSE and R2 in comparison to data set DS-III and DS-IV. Although the proposed model is not among the best fit for DS-III and DS-IV but gives better fit than Kapur et al (2012) model. The proposed model describes the concept of repeat purchase growth in the adopter potential. From the Table 2, it has been observed that there is very low repeat purchase behaviour for the product as expected. As we observed that repeat purchase behaviour is more applicable to consumable product but it can't be ignored for the case of technological consumer durable products.

The loyalty towards a brand is often reflected when the products are bought by the consumer. The adoption of newer version shows consumers' loyalty for the firm's products. To estimate the initial size of the potential population for the new products of the existing firm, measurement of loyal population for the same firm is very important. In such case, the theory of repeat purchasing is exist and the higher is the repeat purchase value, it can be said that the better a firm is doing to keep customers loyal. In this paper, we use the concept of repeat purchase in an innovation and diffusion model based on $I\hat{t}o$ type of stochastic differential equation (SDE). Its applicability and accuracy is illustrated by means of new product sales data. Predictive validity and mean square error have been used to check the validity of the proposed model. It has been shown proposed model performs comparatively better than SDE-based Bass innovation and diffusion model

References

Bass, F.M., A New Product Growth Model For Consumer Durables, *Management Science*, Vol.50, No.12, pp. 1825-1832.2004(2004, reprinted from January 1969)

Bass, F.M., Krishnan, T.V. and Jain, D.C., Why The Bass Model Fits Without Decision Variables, *Marketing Science*, Vol.13, No.3, pp.203-223, 1994.

Easingwood, C., Mahajan, V. and Muller E., A Non Symmetric Responding Logistic Model for Technological Substitution, *Technological Forecasting and Social Change*, Vol.20, pp.199-213, 1981.

Easingwood, C., Mahajan, V. and Muller E., A Non Uniform Influence Innovation Diffusion Model For New Product Acceptance, *Marketing Science*, Vol. 2, pp.273-295, 1983.

Floyd, A Methodology for Trend Forecasting of Figures of Merit, In: J. Bright (ed.), *Technological forecasting for industry and Government: Methods and Applications*, (pp.95-109) Englewood Cliffs, NJ, Prentice Hall.1968

Fisher, J.C. and Pry, R.H., A Simple Substitution Model for Technological Change, *Technological Forecasting and Social Change*, Vol.3, pp. 75-88, 1971.

Kapur P.K., Bardhan A. and Jha P.C., An Alternative Formulation of Innovation Diffusion Model', In: V.K.Kapoor (ed.), *Mathematics and Information Theory*, (pp.17-23) , New Delhi ,Anamaya Publication, 2004.

Kapur P.K., Chaudhary K., Aggarwal, G., and Jha, P.C., On the development of innovation diffusion model using stochastic differential equation incorporating change in the adoption rate, *Int. J. Operational Research*, Vol. 14, No. 4, pp.472-484, 2012.

Lilien, G. L., Kotler, P. and. Moorthy, K. S., *Marketing models*, New Delhi, Printice Hall of India, 1998.

Mahajan, V. and Peterson, R.A., Innovation Diffusion in A Dynamic Potential Adopter Population, *Management Science*, Vol.24, pp.1589-1597, 1978

Mahajan, V., and Peterson, R.A., *Models for Innovation diffusion*, California, Sage Publication, 1985.

Mahajan, V., Muller E. and Bass F.M., New Product Diffusion Model In Marketing: A Review And Direction For Research, *Journal of Marketing*, Vol.54, pp.1-26, 1990.

Oksendal, *Stochastic Differential Equations: An Introduction with Applications*, New York Springer,2003.

Phister, J., *Data Processing Technology and Economics*, Bedford, MA, Santa Monica Publishing Company and Digital Press, 1979.

Rogers, E.M., Diffusion of Innovation, New York, Free Press,1983.

Sharif, M.N., and Ramanathan, K., Polynomial Innovation Diffusion Models, *Technological forecasting and social change*, 20, pp.301-323, 1981.

Schiffman, L.G. and Kanu L.L., Consumer Behaviour, New Delhi, Prentice Hall of India, 1995

Biographies

Kuldeep Chaudhary is currently the Assistant Professor in the Department of Applied Mathematics, Amity Institute of Applied Sciences, Amity University. He obtained his Ph. D. in Operational Research from University of Delhi and M.Sc. from Indian Institute of Technology (IIT), Roorkee respectively. He has published more than 20 research papers in the areas of software reliability, Marketing and Optimization. He has guided M.Sc. Dissertations in Applied mathematics. His research interests include mathematical modelling in optimal control theory and optimization in marketing, Inventory-production and software reliability.

Kalyani Kumar obtained her Master of Sciences in Applied Mathematics from Amity University, Uttar Pradesh

Sunita Mehta Sharma is the Assistant Professor in the Department of Mathematics, Kalindi College, University of Delhi. She has done her M.Sc. from Indian Institute of Technology (IIT), Delhi and M.Phil. in Fuzzy Optimization from the Department of Mathematics, University of Delhi. She is currently pursuing her Ph.D. from Department of Applied Mathematics, Amity Institute of Applied Sciences, Amity University, Uttar Pradesh.