2.1. Entropy of degree distribution

The degree of a node is the number of links that connect the node to other nodes. The degree distribution, g_k, is the probability that a randomly chosen node has *k* links. Valuable information on intrinsic network structure can be inferred from the node degree distribution (Pozrikidis, 2016). As discussed earlier, entropy of degree distribution has been widely used as a measurement of robustness for complex systems. Here this notion is recalled so as to provide helpful clarification.

Let k_i be the degree value of node i and let $\{g_1, g_2, \dots, g_\Delta\}$ define the degree sequence distribution of the network, where g_k is the fraction of nodes with degree k. The entropy of degree distribution of a network, that is, H_d, is a measure of heterogeneity in the network, which can be formulated as:

$$H_d = -\sum_{k=1}^{\Delta} g_k \log_2(g_k) \qquad (2)$$

where Δ represents the maximum degree of nodes in the network. The higher value of H_d denotes more diversity in link distribution. Intuitively, diversity in degree distribution increases the robustness of the network, thus the higher the entropy of the degree distribution, the more robust of the network is. H_d attains its maximum value ($\log_2 \Delta$) when $g_k = \frac{1}{\Delta}$ for any node degree *k*, and its minimum value 0 when the node degrees are either 0 or 1.

Despite a broad use of the degree distribution entropy-based metric in assessing the robustness of networks, H_d only captures very generic topological information of the complex networks and therefore is poorly informative. A reliable robustness assessment of WDNs requires further specification of topological and hydraulic features of the network. Moreover, in WDNs the maximum nodal degree is generally very low because such networks are located on a two-dimensional spaces which are constrained by a number of impediments, therefore the results obtained from the stand alone use of the entropy-based degree distribution metrics might be unreliable (Giustolisi *et al*., 2016).

2.2. Entropy of demand fraction

With the aim of addressing the aforementioned shortcoming of the degree distribution entropy-based metric, the robustness evaluation of WDNs is now extended by proposing a new hydraulic measurement, called *demand fraction entropy-based robustness metric,* (H_q).
Let a network consists of N nodes and let the required flow (demand) at node i be denoted by q_i. For node i, the parameter p_i in Eq. (1) is defined as:

$$p_i = \frac{q_i}{Q_t} \qquad (3)$$

where p_i is demand fraction at node i, and Q_t is the total required flow of the network and given by:

$$Q_t = \sum_{i=1}^{N} q_i \qquad (4)$$

The parameter p_i represents the contribution of node i to the total flow of the network. To develop the demand fraction entropy-based robustness metric, Eq. (1) can be restated as follows:

$$H_q = -\sum_{i=1}^{N} (\frac{q_i}{Q_t}) \log_2(\frac{q_i}{Q_t}) \qquad (5)$$

Since $\sum_{i=1}^{N} p_i$=1, the variables of the set $P = \{p_i: i = 1,2, \dots, n\}$are conditionally reliant on each other. That is, if the demand fraction at a particular node increases, the summation of all other variables tends to decrease and vice versa. From the informational entropy perspective, thus, H_q can be constructed as a measure of the network robustness that indicates the degree of severity of single failure of nodes. The intuitive interpretation of H_q is that a critical node with the highest value of demand fractions contributes more to the drop in overall network performance in response to random failures. In fact, removal of a node with a high demand fraction, leaves a larger number of households without water supply in comparison to the failure of nodes with lower demand fractions. Thus, it is more advantageous from the robustness point of view, if all nodes are of equal value of demand fraction.

2.3. Joint entropy of degree distribution and demand fraction

To achieve a reliable assessment of the robustness, the authors argue that the focus should shift away from an exclusive topological viewpoint or a pure hydraulic approach, towards a combined topological and hydraulic analysis. To this end, the paper proposes a new index drawing on the joint entropy of degree distribution and demand fraction in order to measure the robustness of WDNs.

The joint entropy, $H(X_1, X_2)$, of a pair of random variables with a set of joint probability of $\{p_{ij}: i = 1,2, \dots, N; j = 1,2, \dots, M\}$, is defined as:

$$H(X_1, X_2) = -\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} \log_2(p_{ij}) \le H(X_1) + H(X_2) \quad (6)$$

Demand fraction and degree distribution are stochastically independent variables, hence their joint entropy can be obtained from the following equation (Singh, 2013):

$$H(d, q) = H_d + H_q \quad (7)$$

For a network with N nodes and the maximum degree of Δ, $H(d, q)$ is maximum when $g_i = \frac{1}{\Delta}$ and $q_i = \frac{Q_i}{N}$, hence:

$$H_{max}(d, q) = -(\log_2 \Delta + \log_2 N) \quad (8)$$

The robustness index, RS_I, is defined as the fractional differences between $H_{max}(d, q)$ and $H(d, q)$, which can be expressed by the following equation:

$$RS_I = \frac{H(d, q)}{H_{max}(d, q)} \quad (9)$$

Our robustness index combines both the degree distribution and the demand fraction and can be obtained by substituting Eq. (2), Eq. (5) and Eq. (8) into Eq. (9) as:

$$RS_I = \frac{\sum_{k=1}^{\Delta} g_i \log_2(g_i) + \sum_{i=1}^{N}(\frac{q_i}{Q_t}) \log_2(\frac{q_i}{Q_t})}{\log_2 \Delta + \log_2 N} \quad (10)$$

The robustness index proposed here, is the ratio of the network robustness to the maximum possible robustness, which describes the degree of severity of failures in the network, due to the failure of each individual node. RS_I satisfies the following properties:

1) RS_I is a dimensionless value, which implies the relative entropy.
2) When RS_I is closer to 1 the robustness is higher, and otherwise is smaller.
3) RS_I attains its maximum value ($RS_I = 1$) when $\{g_k = \frac{1}{\Delta}: k = 1,2, \dots, \Delta\}$ and $\{p_i = \frac{Q_t}{N}: i = 1,2, \dots, N\}$.

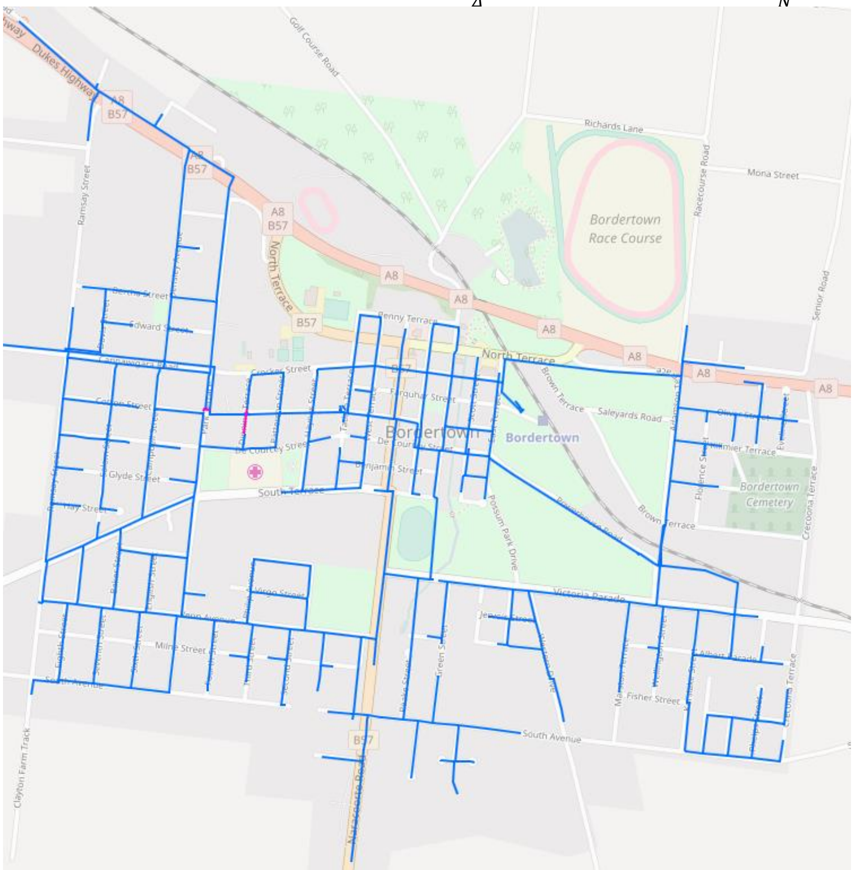


Fig.1. Bordertown water distribution network

3. Application

In order to compute the value of the robustness index, discussed in section 2, it is now time to turn to a real world WDN of Bordertown, an Australian town supplying 2800 inhabitants in the South Australia near the Victorian boarder (Fig. 1). The data for this research has been obtained from the official website of SA Water, containing the layout of the network, pipes diameter, and number of households connected to each pipe (http://sawater.maps.arcgis.com). The example network is mapped into a graph with 216 nodes, representing demand nodes, and 256 edges, representing pipes varying from 80 mm to 375 mm.

3.1. Numerical results

Results from the application of the proposed method to data from the case study are now presented. This subsection starts with computing the entropy of degree distribution (H_d), and then proceeds with obtaining the demand fraction entropy (H_q). The robustness index (RS_I) is finally developed using the values of H_d, H_q, and the maximal entropy.

The parameters used for computing H_d are listed in Table. 1.

Table 1. The parameter used for obtaining H_d

k	n_k	$g_k = \frac{n_k}{N}$	$\log(g_k)$	$g_k \log(g_k)$
1	74	0.3426	-1.5454	-0.5295
2	25	0.1157	-3.1110	-0.3601
3	84	0.3889	-1.3626	-0.5299
4	33	0.1528	-2.7105	-.04141

By substituting the results from Table 1 into Eq. (2), $H_d = 1.8335$ bits.

A similar procedure can be used for calculation of H_q. Due to large number of nodes, Table 2 shows a part of the parameters used for calculating H_q representatively.

Table 2. Some of the parameter used for obtaining H_q

$Node$	$p_i = \frac{q_i}{Q_t}$	$\log(\frac{q_i}{Q_t})$	$\frac{q_i}{Q_t}\log(\frac{q_i}{Q_t})$
1	0.00154	-9.3469	-0.01435
2	0.00154	-9.3469	-0.01435
3	0.00307	-8.3469	-0.02564
4	0.00324	-8.2689	-0.02681
:	:	:	:
105	0.00234	-8.7392	-0.02045
106	0.00586	-7.4153	-0.04344
:	:	:	:
215	0.00341	-8.1949	-0.02797
216	0.00171	-9.1949	-0.01569

By using Eq. (5), $H_q = 7.5604$ bits.

Given $\Delta = 4$ and $N = 216$, now is possible to calculate the robustness index of the example network, using Eq. (10), that is, $RS_I = 0.9630$.

RS_I can be interpreted as a measure of distance from maximum possible entropy of the network. The high value of RS_I in the example network, implies low heterogeneity and subsequently high level of serviceability in the case of random failure of a node.

For completeness, the results of this analysis are summarized in Table 3.

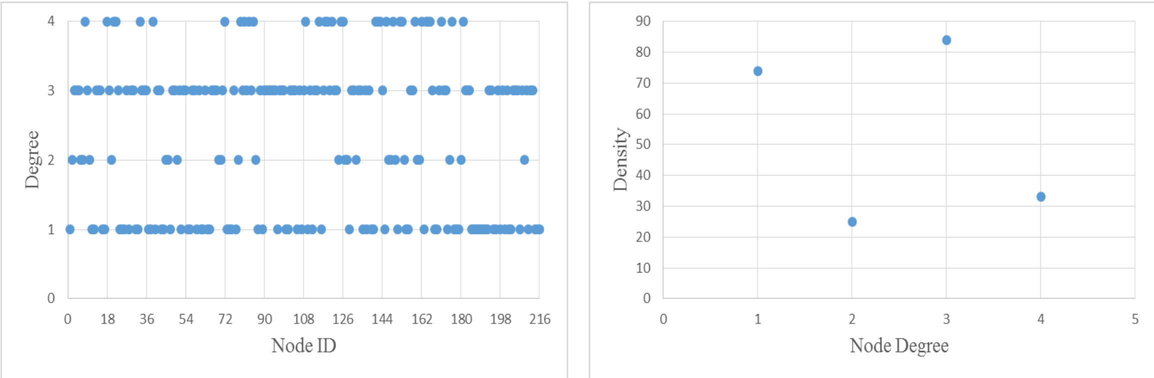
Table 3. Number of nodes, maximum nodal degree, entropy of degree distribution, entropy of demand fraction, robustness index.

N	Δ	H_d	H_q	$H_{max}(d,q)$	RS_I
216	4	1.8335	7.5604	9.7546	0.9630

3.2. Discussion

Comparative discussion on a few key characteristics of nodal degree distribution and demand fraction concepts is presented in this subsection. Visualizations underlying the value of RS_I are also presented here.

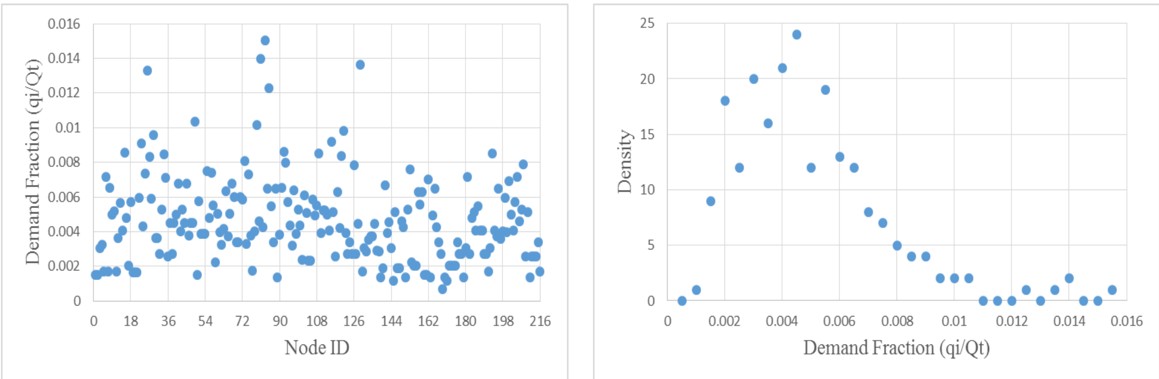
Fig. 2(a) shows the degree of nodes in the example network. A low diversity of the node degrees is observed such that the maximum degree is four. Consequently, only four points can be considered for obtaining the degree distribution entropy. Fig. 2(b) depicts the density against the node degree. The nodes have a maximum of 4 links and a minimum of one link. Degree 3 and 1 nodes are the most occurring degrees in the network, whereas degree 2 and 4 are the less frequent degrees. As expected for a real world WDN, the nodal degree distribution spans over a very limited range from 1 to 4, which implies the low heterogeneity of the network.



(a) (b)

Fig.2. (a) Degree of each node; (b) Nodal degree distribution of the example network

By contrast, as reported by Fig. 3(a), demand fractions span over a wider range of values, ranging from 0.00068 to 0.01501. As discussed in the preceding section, incorporating more variables into the probabilistic model entails a more reliable estimation of robustness. Moreover, as shown in Fig. 3(b), it is observed that the density values are more frequent around the corresponding value of the uniformed demand distribution, that is, $\frac{1}{N} = 0.0046$, which implies homogeneity of the network from the demand distribution perspective.



(a) (b)

Fig.3. (a) Demand fraction of each node; (b) Demand fraction distribution of the case study

The demand fraction can be interpreted as a measure for loss of the network performance when this node is failed. Fig.3 (a) plots the scatterplot of p_i and visually depicts the level of heterogeneity in the network. It can be observed that the loss of performance oscillates between 0.07% and 1.5%. Single failure of 209 out of 216 nodes in the network, results in loss of overall performance at values of less than 1%, thereby confirming low heterogeneity of the studied network.

As above discussion attests, the high value of $RS_I = 0.963$ is not entirely surprising and, indeed this value is in consistent with the fact that the Bordertown WDN is a spatially organized network with low heterogeneity, where most of the nodes have comparable influence form degree and demand perspectives.

4. Conclusion

The method described herein represents a new step toward measuring the robustness of WDNs. The authors advocate the idea that the behavior of a network can be characterized by assessing the influence of each individual node in the network (Singh *et al*., 2015). Different authors have developed topological perspectives to evaluate the robustness of complex networks by assessing the influence of individual nodes. The present research emphasizes that persistent focus on topological properties fails to realistically measure the robustness of WDNs. To remedy this weakness, rooted in entropy theory, a new robustness index is proposed, which combines the topological and hydraulic attributes of the network. The proposed index not only explores the structural heterogeneity of the network by using an entropy-based degree distribution metric, but also incorporates the demand fraction entropy which accounts for the hydraulic attribute of WDNs. This integrated approach yields more useful insights to evaluate the robustness of WDNs in comparison with the conventional approaches.

The paper posits the informational entropy theory as a tool to measure the robustness of WDNs. The work also measures and analyses the robustness of a network from the character of its heterogeneity (Wang *et al*., 2006). On this premises, the joint entropy of degree distribution and demand fraction is employed to measure the network's heterogeneity, which in turn can be interpreted as the measure of robustness. The entropy-based analysis reveals that a uniform demand distributed network with diversity in its degree distribution is less heterogeneous and thus exhibits more robust behavior against random failure of its nodes.

The use of the proposed index was illustrated with a real-world case study. The paper compared the effectiveness of the demand fraction entropy-based metric with the degree distribution entropy-based metric. The numerical results confirmed that the nodal degrees in WDNs span over a small range of values, consequently the degree distribution entropy-based metric is not informative enough and therefore stand-alone use of this metric hinders a reliable assessment of the robustness in WDNs (Giustolis *et al*., 2017).

Though the degree distribution and demand fraction are good indicators of topological and hydraulic (respectively) performance of WDNs, these metrics don't entirely characterize the structural and hydraulic behaviors of the network. Further research can build upon our current study by incorporating more topological (e.g. centrality measures) as well as hydraulic (e.g. pressure head at nodes) metrics into our method. Furthermore, since nodal demands are one of the main sources of uncertainty in WDNs (Sanz and Perez, 2014), it would be helpful if future research could engage in reducing the uncertainty in model outputs induced by uncertainty in nodal demands.

The authors believe that the robustness evaluation of WDNs requires incorporating more factors than is presented in this paper and hope the current study further encourages researchers to develop more innovative methods.

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