Optimal Ordering Policy for non-instantaneous deteriorating items with partially backlogged and stock dependent demand under two-warehouse storage facility

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Abstract

This paper develops a two warehouse inventory model for non-instantaneous deteriorating items with partial backlogging and stock-dependent demand. In the model, shortages are allowed and the backlogging rate is variable and dependent on the waiting time for the next replenishment. The necessary and sufficient conditions of the existence and uniqueness of the optimal solution are shown. Numerical example is presented to demonstrate the developed model, followed by the sensitivity analysis of the optimal solution with respect to key parameters of the inventory system. The purpose of this study is to determine the optimal replenishment policy by minimizing the total optimal cost. The proposed model can assist a decision maker in making important replenishment decisions.

Keywords
Inventory; non-instantaneous deterioration; two-warehouse; partial backlogging; Stock-dependent demand

1. Introduction and Literature Review

Generally, deterioration is defined as damage, spoilage, decay, obsolescence, evaporation, pilferage, etc., that result in decreasing the usefulness of the original one. For items such as steel, hardware, glassware, and toys, the rate of deterioration is low, so there is little need for considering deterioration in the determination of the economic lot size. However, some items such as food items, pharmaceuticals, chemicals, blood, alcohol, gasoline, and radioactive chemicals, etc. deteriorate rapidly over time. Thus, the loss from deterioration should not be ignored. Ghare and Schrader [1] presented an EOQ model for deteriorating items assuming exponential decay. Covert and Phillip [2] extended Ghare and Schrader’s [1] model with the assumption of Weibull distribution deterioration. Two surveys on trends in modeling of continuously deteriorating inventory are presented by Raafat et al [3], Goyal & Giri [4] and Bakker et al. [5]. In current scenario, two warehouse inventory models are followed to avoid frequent transportation inconvenience, to avail the advantage of price concession, trade credit and to guard against the scarcity of the commodity. Thus when large quantity of goods cannot be stored in own warehouse (OW), with limited capacity, rented warehouse (RW) is hired. In recent years good number of two warehouse inventory models has been discussed by several researchers. Chung and Huang [6], Liang and Zhou [7], Bhunia et al. [8] developed two warehouse inventory models for deteriorating items considering permissible delay in payments. Continuing the research ahead, a number of research papers in this interesting area have been published by several researches over the last few decades. One may refer to the recent works of Zhou and Yang [9], Lee [10], Yang [11, 12, 13], Hsieh et al. [14], Niu & Xie, [15], Lee & Hsu [16], Liao et al. [17], Jaggi et al. [18, 19, 20] and Tiwari et al. [21].

Incorporating the above facts, this paper develops a two warehouse inventory model for non-instantaneous deteriorating items with partial backlogging and stock-dependent demand. In the model, shortages are allowed and the backlogging rate is variable and dependent on the waiting time for the next replenishment. The necessary and sufficient conditions of the existence and uniqueness of the optimal solution are shown. Numerical example is presented to demonstrate the developed model, followed by the sensitivity analysis of the optimal solution with respect to key parameters of the inventory system. The purpose of this study is to determine the optimal replenishment policy by minimizing the total optimal cost. The proposed model can assist a decision maker in making important replenishment decisions.
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2. Assumption and Notations
The mathematical models of the two-warehouse inventory problems are based on the following assumptions:

1. The own warehouse (OW) has a fixed capacity of W units; the rented warehouse (RW) has unlimited capacity.
2. The inventory costs (including holding cost and deterioration cost) in the RW are higher than those in OW.
3. Replenishment rate is instantaneous.
4. Lead-time is negligible.
5. Demand rate \( D(t) \) is a function of stock level \( I(t) \) at time \( t \) which is given by
   \[
   D(t) = \begin{cases} 
   a + bI(t) & : I(t) > 0 \\
   a & : I(t) < 0 
   \end{cases},
   \]
   where \( a, b \) are positive constants.
6. The planning horizon of the inventory system is infinite.
7. Unsatisfied demand/shortages are allowed. Unsatisfied demand is partially backlogged and the fraction of shortages backlogged is a differentiable and decreasing function of time \( t \), denoted by \( g(t) \), where \( t \) is the waiting time up to the next replenishment. We have defined the partial backlogging rate \( g(t) = e^{-\delta t} \), where \( \delta \) is a positive constant.

In addition, the following notations are used throughout this paper.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>replenishment cost per order</td>
</tr>
<tr>
<td>( c )</td>
<td>purchasing cost per unit</td>
</tr>
<tr>
<td>( W )</td>
<td>capacity of the owned warehouse</td>
</tr>
<tr>
<td>( Q )</td>
<td>order quantity per cycle</td>
</tr>
<tr>
<td>( Z )</td>
<td>maximum inventory level per cycle</td>
</tr>
<tr>
<td>( H )</td>
<td>holding cost per unit per unit time in OW</td>
</tr>
<tr>
<td>( F )</td>
<td>holding cost per unit per unit time in RW, where ( F &gt; H )</td>
</tr>
<tr>
<td>( s )</td>
<td>the backlogging cost per unit per unit time, if shortage is backlogged.</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>unit opportunity cost due to lost sale, if the shortage is lost.</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>deterioration rate in OW, where ( 0 \leq \alpha &lt; 1 )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>deterioration rate in RW, where ( 0 \leq \beta &lt; 1; \alpha &lt; \beta )</td>
</tr>
<tr>
<td>( t_d )</td>
<td>time period during which no deterioration occurs.</td>
</tr>
<tr>
<td>( t_r )</td>
<td>time at which the inventory level reaches zero in RW.</td>
</tr>
<tr>
<td>( t_w )</td>
<td>time at which the inventory level reaches zero in OW.</td>
</tr>
<tr>
<td>( T )</td>
<td>the length of the replenishment cycle in year.</td>
</tr>
<tr>
<td>( I_0(t) )</td>
<td>inventory level in the OW at any time ( t ) where ( 0 \leq t \leq T )</td>
</tr>
<tr>
<td>( I_r(t) )</td>
<td>inventory level in the RW at any time ( t ) where ( 0 \leq t \leq T )</td>
</tr>
<tr>
<td>( TC_i )</td>
<td>present value of total relevant cost per unit time for case ( i=1, 2 )</td>
</tr>
<tr>
<td>( B(t) )</td>
<td>backlogged level at any time ( t ) where ( t_w \leq t \leq T )</td>
</tr>
<tr>
<td>( L(t) )</td>
<td>number of lost sales at any time ( t ) where ( t_w \leq t \leq T )</td>
</tr>
</tbody>
</table>

3. Model Formulation
This section formulates the replenishment problem of a two warehousing inventory model for a single non-instantaneous deteriorating item with complete backlogging. Initially, a lot size of \( Q \) units enters in the system. After meeting the backorders, \( Z \) units left in the inventory system, out of which \( W \) units are kept in OW and the remaining \((Z-W)\) units are kept in the RW. As the deterioration of item is non-instantaneous, so initially, the units do not deteriorate for the time period \((t_d)\) and after that the deterioration begins. Broadly, there can be two cases: when \( t_d \) (time during which no deterioration occurs) is less than \( t_r \) (time during which inventory in RW becomes zero) and when \( t_d \) is greater than \( t_r \).
Case 1: When $t_d < t_r$

During the time interval $[0, t_d]$, there is no deterioration, the inventory in RW is depleted only due to demand whereas in OW inventory level remains the same. In the time interval $[t_d, t_r]$ the inventory level in RW is dropping to zero due to the combined effect of demand and deterioration and in OW inventory depleted due to deterioration only. Further, during the time interval $[t_r, t_w]$ depletion of inventory occurs in OW due to the combined effect of demand and deterioration and it reaches to zero at time $t_w$. Moreover, during the interval $[t_w, T]$, the demand is backlogged. The behavior of the model over the whole cycle $[0, T]$ has been graphically represented below in Figure 1.

![Two-warehouse inventory system when $t_d < t_r$](image)

**Figure 1:** Two-warehouse inventory system when $t_d < t_r$

Therefore, the differential equations that describe the inventory level in the RW and OW at any time $t$ over the period $(0, T)$ are given by:

$$\frac{dI_r(t)}{dt} = -\left( a + bI_r(t) \right), \quad 0 \leq t \leq t_d$$  \hspace{1cm} (1)

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -\left( a + bI_r(t) \right), \quad t_d < t \leq t_r$$  \hspace{1cm} (2)

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = 0, \quad t_d < t \leq t_r$$  \hspace{1cm} (3)

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = -\left( a + bI_0(t) \right), \quad t_r < t \leq t_w$$  \hspace{1cm} (4)

$$\frac{dB(t)}{dt} = a e^{-\delta(t-t_0)}, \quad t_w < t \leq T$$  \hspace{1cm} (5)

The solutions of the above five differential equations (1), (2), (3), (4) and (5) with boundary conditions $I_r(0) = Z - W, I_r(t_r) = 0, I_0(t_d) = W, I_0(t_w) = 0$ & $B(t_w) = 0$ respectively are

$$I_r(t) = \left( Z - W \right) e^{-bt} + \frac{a}{b} \left( e^{-bt} - 1 \right), \quad 0 \leq t \leq t_d$$  \hspace{1cm} (6)

$$I_r(t) = \frac{a}{\beta + b} \left( e^{(\beta + b)(t - t_r)} - 1 \right), \quad t_d < t \leq t_r$$  \hspace{1cm} (7)
The total cost per cycle consists of the following elements:

1. The replenishment cost is $A$
2. The inventory holding cost in RW

\[
J(T) = \begin{cases} 
F \left( \int_{t_0}^{t_1} I_r(t) \, dt + \int_{t_1}^{t_2} I_s(t) \, dt \right) \\
F \left( \frac{(Z - W)}{b} \left(1 - e^{-b t_2} \right) + \frac{a}{b} \left(1 - e^{-b t_2} - b t_2 \right) + \frac{a}{(b + \beta)} \left\{ e^{(\beta + b)(t_2 - t_1)} - 1 \right\} \right) 
\end{cases}
\]

3. The inventory holding cost in OW

\[
J(T) = \begin{cases} 
H \left( \int_{t_0}^{t_1} W dt + \int_{t_1}^{t_2} I_0(t) \, dt + \int_{t_2}^{t_3} I_0(t) \, dt \right) \\
H \left( \frac{W}{\alpha} \left(1 - e^{-\alpha t_3} \right) + \frac{a}{(\alpha + \beta)} \left\{ e^{(\alpha + \beta)(t_3 - t_2)} - 1 \right\} \right) 
\end{cases}
\]

Therefore, the order quantity over the replenishment cycle can be determined as $Q = Z + B(T)$ (Using Equations (13) and (15))

\[
Q = W + \frac{a}{(b + \beta)} \left\{ e^{(\beta + b)(t_2 - t_1)} - 1 \right\} e^{b t_2} - \frac{a}{b} \left(1 - e^{b t_2} \right) + \frac{a}{(b + \beta)} \left(1 - e^{-\beta (T - t_2)} \right) 
\]
4. The backlogging cost is
\[ s \int_{t_i}^{T} B(t) \, dt \]
\[ = \frac{sa}{s} \left[ \frac{1}{s} - e^{-s(T-t_i)} - (T-t_w) e^{-s(T-t_w)} \right] \]

5. The opportunity cost due to lost sale is
\[ c_2 \int_{t_i}^{T} \left[ 1 - e^{-\delta(T-t)} \right] \, dt \]
\[ = c_2 \left[ T - t_w - \frac{1}{\delta} \left( 1 - e^{-\delta(T-t_w)} \right) \right] \]

6. The deterioration cost is
\[ c \left[ \beta \int_{t_i}^{T} l_r(t) \, dt + \alpha \int_{t_i}^{T} l_0(t) \, dt \right] \]
\[ = c \left\{ \frac{a}{(\beta + b)} \left[ e^{(\beta + b)(t_i - t_u)} - 1 \right] + \left( T - t_r \right) \right\} + \frac{a}{(\alpha + b)} \left[ e^{(\alpha + b)(t_u - t_r)} - 1 \right] \]
Hence, using the above elements, the total relevant cost per unit time is given by
\[ T\text{C}_1(t_i, T) = \frac{1}{T} \left( A + F \left( Z - W \right) \right) + \frac{a}{\beta + b} \left[ 1 - e^{-b(t_i - t)} \right] + \frac{a}{(\beta + b)} \left[ e^{(\beta + b)(t_i - t_w)} - 1 \right] \]
\[ + \frac{a}{(\alpha + b)} \left[ e^{(\alpha + b)(t_u - t_i)} - 1 \right] \]
\[ + \left( T - t_r \right) \]
\[ + \frac{sa}{\delta} \left[ 1 - e^{-\delta(T-t_w)} \right] - (T - t_w) e^{-\delta(T-t_w)} \]
\[ + c_2 \left[ T - t_w - \frac{1}{\delta} \left( 1 - e^{-\delta(T-t_w)} \right) \right] \]
\[ + c \left\{ \frac{a}{(\beta + b)} \left[ e^{(\beta + b)(t_i - t_u)} - 1 \right] + \left( T - t_r \right) \right\} + \frac{a}{(\alpha + b)} \left[ e^{(\alpha + b)(t_u - t_i)} - 1 \right] \]
\[ + \left( T - t_r \right) \]

where
\[ Z = W + \frac{a}{\beta + b} \left[ e^{(\beta + b)(t_i - t_u)} \right] - \frac{a}{b} \left[ 1 - e^{b(t_i - t_u)} \right] \]
and
\[ t_w = t_r + \frac{1}{a} \ln \left( \frac{a + \beta}{a} \right) \]

\[ \text{Case 2. When } t_i > t_r \]
In this case, time during which no deterioration occurs is greater than the time during which inventory in RW becomes zero and the behavior of the model over the whole cycle \([0, T]\) has been graphically represented as in Figure 2
The differential equations that describe the inventory level in the RW and OW at any time $t$ over the period $(0, T)$ are given by:

\[
\frac{dI_r(t)}{dt} = -(a + bI_r(t)), \quad 0 \leq t \leq t_r \tag{18}
\]

\[
\frac{dI_0(t)}{dt} = -(a + bI_0(t)), \quad t_r < t \leq t_d \tag{19}
\]

\[
\frac{dI_0(t)}{dt} + aI_0(t) = -(a + bI_0(t)), \quad t_d < t \leq t_w \tag{20}
\]

\[
\frac{dB(t)}{dt} = ae^{-\delta(T-t)}, \quad t_w < t \leq T \tag{21}
\]

The solutions of the above four differential equations (18), (19), (20) and (21) with boundary conditions

\[I_r(t_r) = 0, I_0(t_r) = W, I_0(t_w) = 0, B(t_w) = 0 \]

respectively are

\[
I_r(t) = \frac{a}{b} \left( e^{b(t_r-t)} - 1 \right), \quad 0 \leq t \leq t_r \tag{22}
\]

\[
I_0(t) = \left( W + \frac{a}{b} \right) e^{b(t_r-t)} - \frac{a}{b}, \quad t_r < t \leq t_d \tag{23}
\]

\[
I_0(t) = \frac{a}{\alpha + b} \left( e^{(\alpha + b)(t_r-t)} - 1 \right), \quad t_d < t \leq t_w \tag{24}
\]

\[
B(t) = \frac{a}{\delta} \left( e^{-\delta(T-t)} - e^{-\delta(T-t_r)} \right); \quad t_w < t \leq T \tag{25}
\]

The number of lost sales at time $t$ is

\[
L(t) = \int_{t_r}^{t} a \left[ 1 - e^{-\delta(T-t)} \right] dt; \quad t_w < t \leq T
\]

\[
= a \left[ (t-t_w) - \frac{1}{\delta} \left( e^{-\delta(T-t)} - e^{-\delta(T-t_r)} \right) \right] \tag{26}
\]

Considering continuity of $I_0(t)$ at $t = t_d$, it follows from Equations (23) and (24) that

\[
\left( W + \frac{a}{b} \right) e^{(\alpha + b)(t_r-t_d)} - \frac{a}{b} = \frac{a}{\alpha + b} \left( e^{(\alpha + b)(t_r-t)} - 1 \right) \tag{27}
\]
\[ t_w = t_d + \frac{1}{a + b} \ln \left[ 1 + \frac{a + b}{a} \left\{ \left( W + \frac{a}{b} e^{b(t_r - t_d)} - \frac{a}{b} \right) \right\} \right] \] 

(28)

Now, at \( t = 0 \) when \( I_r(t) = Z - W \) and solving Equation (22) we get the maximum inventory as

\[ Z = W + \frac{a}{b} \left( e^{b(t_r - t_d)} - 1 \right) \] 

(29)

Putting \( t = T \) in Equation (25), the maximum amount of demand backlogged per cycle is

\[ B(T) = \frac{a}{\delta} \left( 1 - e^{-\delta(T - t_r)} \right) \] 

(30)

Therefore, order quantity is

\[ Q = Z + B(T) \]

(Using Equations (29) and (30))

(31)

Again, the total cost per cycle consists of the following elements:

1. The replenishment cost = \( A \)

2. The inventory holding cost in RW

\[ = F_a \frac{e^{b(t_r - t_d)} - 1}{b - t_r} \]

3. The inventory holding cost in OW

\[ = H \left( \frac{t_r - t_d}{0} \int W(t) dt + \int I_b(t) dt + \int I_w(t) dt \right) \]

\[ = H \left( Wt_r + \left( W + \frac{a}{b} e^{b(t_r - t_d)} - \frac{a}{b} (t_d - t_r) + \frac{a}{(a + b)} \left( e^{(a+b)t_r} - e^{-t_r(1+b)} \right) - (t_r - t_d) \right) \right) \]

4. The backlogging cost

\[ = \frac{s a}{\delta} \left( \frac{1}{\delta} \left( 1 - e^{-\delta(T - t_r)} \right) - (T - t_w) e^{-\delta(T - t_r)} \right) \]

5. The opportunity cost due to lost sale

\[ = c_1 \int_{t_r}^{T} \left( 1 - e^{-\delta(T - t)} \right) D dt \]

\[ = c_1 a \left( T - t_w - \frac{1}{\delta} \left( 1 - e^{-\delta(T - t_r)} \right) \right) \]

6. The cost for deteriorated items

\[ = \frac{c a}{(a + b)} \left( e^{(a+b)t_r} - e^{-t_r(1+b)} \left( 1 - t_r - t_d \right) \right) \]

Hence, the total relevant cost per unit time during the cycle \((0, T)\) is given by
The present worth of the total relevant cost per unit time during the cycle \((0, T)\) is given by

\[
TC_1(t_r, T) = \begin{cases} 
W + \frac{a}{b} \left( e^{rt_r} - 1 \right) & \text{if } t_d \leq t_r \\
\frac{a}{b} \left( e^{rt_d} - e^{rt_r} \right) & \text{if } t_d \geq t_r 
\end{cases}
\]

(33)

which is a function of two continuous variable \(t_r\) and \(T\).

**Optimality:**

Our problem is to determine the optimum value of \(t_r\) and \(T\) which minimizes \(TC(t_r, T)\). The necessary conditions for minimization of the total cost function given by Equation (33) are

\[
\frac{\partial TC_i(t_r, T)}{\partial t_r} = 0, \text{ and } \frac{\partial TC_i(t_r, T)}{\partial T} = 0 \quad \text{for } i = 1, 2 
\]

(34)

Equation (34) can be solved simultaneously for the optimal values of \(t_r\) and \(T\) (say \(t_r^*\) and \(T^*\)) for \(i = 1, 2\) provided, it also satisfies the following sufficient conditions.

\[
\frac{\partial^2 TC_i(t_r, T)}{\partial t_r^2} > 0, \quad \frac{\partial^2 TC_i(t_r, T)}{\partial T^2} > 0 \quad \text{and } \quad \left[ \frac{\partial^2 TC_i}{\partial t_r^2} \frac{\partial^2 TC_i}{\partial T^2} - \frac{\partial^2 TC_i}{\partial T \partial t_r} \frac{\partial^2 TC_i}{\partial t_r \partial T} \right] > 0 \quad \text{for } i = 1, 2
\]

Mathematically, it is very difficult to prove the sufficient conditions, so convexities of cost function for both cases are shown graphically in figure 3.

**Figure 3:** Convexity of cost function w.r.t. \(t_r\) and \(T\)

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**Case 1:** When \(t_d < t_r\)

**Case 2:** When \(t_d > t_r\)
4. Numerical and Sensitivity analysis

Example: To illustrate the results, let us consider an inventory system with the following data: $A = $250/order, $s = $10/unit/year, $c = $20/unit, $c_1 = $5/unit, $H = $0.5/unit/year, $F = $0.7/unit/year, $W = 200$units, $a = 80$, $b = 10$, $a = 0.05/unit$, $\beta = 0.03/unit$, $R = 0.06$, $\delta = 0.9$, $t_0 = 0.1/year$.

The optimal solutions for given example is shown in table 1 of our proposed inventory model.

Table 1: Solution of Example

<table>
<thead>
<tr>
<th>Case</th>
<th>$t_r$</th>
<th>$t_0$</th>
<th>$T$</th>
<th>$Z^*$</th>
<th>$B^*$</th>
<th>$Q^*$</th>
<th>$TC^*(t_0, T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.4247</td>
<td>2.5406</td>
<td>331.32</td>
<td>75.65</td>
<td>406.97</td>
<td>469.07</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>0.4247</td>
<td>2.5597</td>
<td>213.74</td>
<td>76.87</td>
<td>290.61</td>
<td>467.32</td>
</tr>
</tbody>
</table>

5. Conclusion and Future Scope

In this study, a two-warehouse inventory model for non-instantaneous deteriorating items with stock-dependent demand has been developed. Shortages are allowed and partially backlogged. The deterioration rate in OW is assumed to be higher than in RW, and the holding cost in RW is greater than that in OW because of the better storage facilities provided in RW. The items stored have the property of non-instantaneous deterioration. A mathematical model has been developed to determine the optimal ordering policy which minimizes the present worth of total optimal cost. The study concludes with numerical examples. For future research, the present model can be formulated by considering permissible delay in payments, and two-level trade credit policy also can be incorporated. To be more realistic, the model can be extended with different types of variable demand such as stock and price-dependent demand, credit linked demand and advertisement dependent demand, etc.

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References


**Biography**

**Chandra K Jaggi**, Professor, Department of Operational Research, University of Delhi, India. He is Fellow Member of International Science Congress Association since 2012 and in 2009 was awarded Certificate for his Significant Contributions in Operation Management by The Society of Reliability Engineering, Quality and Operations Management, New Delhi. He was awarded Shiksha Rattan Puraskar (for Meritorious Services, Outstanding Performance and Remarkable Role) in 2007 by India International Friendship Society. His research interest lies in the field of Supply Chain and Inventory Management. He has publications papers in various international/national journals including Journal of Operational Research Society, U.K., European Journal of Operational Research, International Journal of Production Economics, International Journal of Systems Sciences, Applied Mathematics and Computation, Applied Mathematical Modelling, Annals of Operations Research, Computers and Industrial Engineering, Applied Mathematics & Information Sciences and Scientia Iranica, Transaction E: Industrial Engineering. etc. He is a reviewer of more than 25 International/National journals. He is Editor-in-Chief of IJICM and Associate Editor of IJSAEM, Springer, Co-Editor / Reviewer-In-Charge of The Gsrf JMSOR and on the Editorial Board of the IJSS: Operations & Logistics, IJSOI, AJOR, IJECBS, JASR, AJBAS. He has travelled extensively in India and abroad and delivered Key note and invited talks.