Multi-objective economic-statistical design of a new t-Chart based on the process capability index

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Abstract

Control charts, as important tools for the statistical process control, play a vital role to improve the efficiency of the industrial processes. In various fields, the monitored quality characteristic follows a non-normal distribution. Considering the assumption that the time between events follows the exponential distribution, a new t-chart which is based on the process capability index (PCI) is developed in this paper. A multi-objective economic-statistical model is proposed to optimally determine the design parameters of a t-control chart. An algorithm using data envelopment analysis (DEA) is employed to solve the proposed model. The procedure of the algorithm is discussed with the help of a numerical example.

Keywords
Exponential distribution; t control chart; Process Capability Index (PCI); multi-objective economic-statistical design; Data Envelopment Analysis (DEA)

1. Introduction

Statistical Process Control (SPC) is an industry-standard methodology for measuring, controlling, and improving the quality and productivity of manufacturing and service enterprises. A Control chart, among seven basic tools of SPC, is a chronological graph of quality data obtained from a process which attempts to differentiate “assignable” sources of variation from “common” sources. Proper analyzing of the behavior of a process using the appropriate control chart will lead to maintain the process in the in-control state and improve it through reduction in the variability. In addition, minimizations in the cost of inspection and nonconforming products are expected (Montgomery, 2008).

The control charts to monitor the continuous data are frequently developed by considering the normal assumption for the distribution of the interested quality data collected in subgroups to utilize the central limit theorem. In reality, these assumptions may not be confirmed for the data follow a non-normal distribution or have been collected based on the individual set. Thus, using this type of control charts may bring problems interrupting the results. The exponential distribution, as a non-normal distribution, is usually well fitted to highly skewed data such as waiting times and the time between events (Santiago and Smith, 2013). This kind of control charts is called the t-chart, which is well investigated by: Mohammad (2004), Mohammad and Laney (2006), Santiago and Smith (2013), Aslam et al. (2014), Aslam et al (2015), and Azam et al. (2015).

Process capability indices (PCIs) are used to evaluate whether the process is capable within the given specification limits or not. Detailed discussions of PCIs and their properties are presented in Kotz and Lovelace (1998). Subramani and Balamurali (2012) designed the capability-based variable control chart which combines the two stage processes into a single stage process for an on-line process control. Ahmad et al. (2014) extended the proposed
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design for the X-bar chart, developed by Subramani and Balamurali (2012), under the repetitive sampling scheme. More recently, Aslam et al. (2016) designed a new t-chart using the process capability index ($C_{pk}$). This Index, which considers both magnitude of process variance and departure from midpoint of specifications limits, is given by:

$$C_{pk} = \min \left\{ \frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma} \right\},$$

where $USL$ and $LSL$ are upper and lower specification limits, respectively; $\mu$ is average of process; and $\sigma$ is process standard deviation. In practice, the unknown values of $\mu$ and $\sigma$ can be estimated by $\bar{X} = 1/n \sum_{i=1}^{n} X_i$ and

$$S = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}}.$$

In order to implement a control chart, it is of great importance to search for the best design parameters. Generally, the design of a control chart refers to the specification of three parameters, namely, sample size ($n$), the time interval between succeeding samples (sampling frequency, $h$), and the control limits width ($k$) in terms of standard deviation (Mobin et al., 2015; Tavana et al., 2016). By tradition, the effectiveness of control charts is evaluated without consideration of economic criterion. In an attempt to make improvement, the economic (cost-effective) design of control charts aroused interest among scholars to explore the design parameters that minimize the costs related to the process monitoring. Numerous researches, motivated by that of Duncan (1956), have investigated the economic design of control charts using a single objective formulation. Nevertheless, the major weakness of pure economic design for overlooking the statistical properties was criticized by Woodall (1986). Thus, Saniga (1989) introduced an economic-statistical model that minimized the Duncan’s cost based design subject to some statistical constraints. Since the importance of statistical properties is the same as economics, this approach seems ineffective and therefore, simultaneous optimization of both properties has been considered by researchers (Mobin et al., 2015; Tavana et al., 2016).

In the area of multi-objective problems, many authors have considered simultaneous optimization of all objectives in their studies. Chen and Liao (2004) proposed a procedure to design X control chart in which the efficient solutions were selected from obtained Pareto optimal set by means of data envelopment analysis (DEA). Asadzadeh and Khoshalhan (2009) employed a similar procedure for multiple assignable causes. Bashiri et al. (2013) and Amiri and Jafari-Namin (2015) have implemented Chen and Liao’s (2004) procedure for the design of the np-chart and different c-chart respectively. Mobin et al. (2015) applied a genetic algorithm enhanced with DEA to optimize the multi-objective design of an X-bar control chart. More recently, Tavana et al. (2016) proposed an integrated approach, which is based on multi-objective evolutionary algorithms combined with multiple criteria decision making methods, to optimize the multi-objective design of X-bar control chart.

DEA is a powerful optimization approach to evaluate the relative efficiency of decision making units (DMUs) with multiple inputs and outputs. Charnes et al. (1978) developed DEA via generalization of the Farrell’s single input, single output efficiency measurement. Two main reasons that make DEA more attractive are: 1) the general and supple definition of a DMU, and 2) fairly few assumptions involved in the modeling. The first DEA approach gained a lot of attention is known as the CCR model. In the CCR mathematical programming model, the performance of a specific DMU is assessed with respect to the performance of the remaining DMUs.

In this paper, a multi-objective economic-statistical design of the new t-chart using the process capability index by defining proper DMUs is proposed. To the best of our knowledge, there is no such work available in the literature. The rest of the paper is organized as follows: The new t-chart based on process capability index is introduced in the next section. In Section 3, the multi-objective model is presented. A brief description of the data envelopment analysis approach is given in section 4. Then, an algorithm using the DEA is employed with some modifications to solve the model. Section 6 includes a numerical example to illustrate the solution procedure. Some concluding remarks are given in the last section.

2. The new t-chart based on process capability index

Suppose that a quality characteristic denoted by $T$ follows the exponential distribution with the following probability density function:

$$f(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}}, \quad t > 0,$$

where $\theta$ is the scale parameter.
where $\theta$ is the mean of the exponential distribution. Consider $\theta_0$ be the in-control mean and $\theta_1$ be the mean of the shifted process. Aslam et al. (2016) used the transformed variable $T^* = T^{1/3.6}$ to convert the exponential data to an approximate normal distribution so that symmetric limits can be developed for the control charts. The modification of $C_{pk}$ using the transformation is given as:

$$
\hat{C}_{pkT^*} = \min \left\{ \frac{USL^* - T^*}{3S_T^*}, \frac{\bar{T}^* - LSL^*}{3S_T^*} \right\},
$$

where $USL^*$ and $LSL^*$ are respectively the transformed specification limits, $\bar{T}^*$ is the sample mean and $S_T^*$ is the sample standard deviation of the transformed data.

Therefore, the proposed control chart by Aslam et al. (2016) is stated as follows

Step 1: Select a random sample of size $n$ items, measure its quality characteristic $T$ and transform to $T^* = T^{1/3.6}$,

Step 2: Calculate $\hat{C}_{pkT^*}$ using (3),

Step 3: Compute the upper control limit (UCL) and lower control limit (LCL) for the new t-chart as follows

$$
\text{UCL} = E(\hat{C}_{pkT^*}) + k \sqrt{Var(\hat{C}_{pkT^*})},
$$

$$
\text{LCL} = E(\hat{C}_{pkT^*}) - k \sqrt{Var(\hat{C}_{pkT^*})}.
$$

3. The Proposed mathematical model

In this section, after making some assumptions about the model, the economic cost function is created. The proposed model is presented in the following sub-sections.

3.1 Assumptions

In order to simplify the mathematical manipulation and analysis, the following assumptions are considered to be hold:

1. The transformed quality characteristic, originally distributed exponentially, follows a Normal distribution,
2. The process is either in-control or out-of-control and is initially in the ‘in-control’ state; that is, $\theta = \theta_0$,
3. When a random assignable cause of magnitude $\delta$ occurs, leads the process mean to shift from $\theta_0$ to $\theta_1 = \theta_0 + \delta\theta_0$,
4. The occurrence of an assignable cause possesses an exponential distribution with mean time $1/\lambda$,
5. The process is allowed to continue during the search and repair.

3.2 The economic cost function

According to Duncan’s definition (1956) of the expected hourly cost, an expected cycle length and an expected cost of the cycle can be formulated in economic model construction. As a ratio between the expected cost during a cycle and the expected cycle time length, the expected hourly cost is adapted to the new t-chart as follows:

$$
E_h = \frac{f + vn + \lambda(W + YA + MB)}{1 + \lambda B},
$$

where $f$ is the fixed cost of sampling an inspection unit, $v$ is the variable cost of sampling an inspection unit, $W$ is the average cost to detect an assignable cause, $Y$ is the cost of verifying a false alarm, $M$ is the hourly loss due to poor quality of units, $A$ is the average number of false alarms per cycle, $B$ is the average time of the process being in out-of-control state, $r$ is the average time of occurrence of an assignable cause between samples, $g$ is the time required to sample, inspect and interpret the results, and $D$ is the time to discover and repair the assignable cause.
3.3 Multi-objective design of the new t-chart

In addition to statistical perspective, designing a control chart has several economic consequences as presented in Amiri and Jafarian-Namin (2015) as follows:

$\min \ E_L(S)$
$\max \ ATS_0(S)$
$\min \ ATS_1(S)$

s.t. \ $E_L \leq E_L^U,$ \ \ \ \ \ \ \ \ (7)
$ATS_0 \geq ATS_0^L$
$ATS_1 \leq ATS_1^U$

where, $E_L$ is expected hourly cost, $S=(n, h, k)$ is a possible set of design parameters, $ATS_0$ is the average time to signal when a false alarm occurs, and $ATS_1$ is the average time to signal when an assignable cause occurs. In addition, $E_L^U, ATS_0^U$ and $ATS_1^U$ are the desired bounds determined by decision maker.

3.3.1 Statistical performances of the new t-chart

The statistical performance of the new t-chart will be evaluated on the basis of average time to signal (ATS). Thus, it is necessary to calculate $\alpha$ (probability of false alarm) and $P$ (detection power) in order to obtain $ATS_0$ and $ATS_1$ respectively as mentioned above. However, the derivations of such probabilities are challenging. Thus, Monte Carlo simulation is used to measure the statistical performances of the new t-chart as follows (Aslam et al., 2016):

1. Sample mean and variance of control statistic
   1.1. Generate 1,000 subgroups of a random sample of size $n$ at each subgroup from the exponential distribution being in the in-control state with specified parameter.
   1.2. Calculate $\hat{C}_{pkT}^*$ for each subgroup.
   1.3. Calculate $E(\hat{C}_{pkT}^*)$ and $\sqrt{Var(\hat{C}_{pkT}^*)}$ from 1,000 subgroups.

2. Setting up control limits
   2.1. Select the initial values of $k$
   2.2. Generate a random variable at each subgroup from the exponential distribution being in the in-control state with specified parameter.
   2.3. Calculate $\hat{C}_{pkT}^*$ for $i^{th}$ subgroup.
   2.4. If the process is declared as in-control, go to Step 2.5. Otherwise, record the number of subgroups so far as the in-control run length.
   2.5. Repeat Steps 2.2 through 2.4 for 10,000 times to calculate $0/ARL_0$, and then $\alpha = 1/ARL_0$.

3. Evaluating the out-of-control ARL
   3.1. Generate a random sample of size $n$ for a subgroup from the exponential distribution considering a mean shift of $\theta_1 = \theta_0 + \delta \theta_0$.
   3.2. Calculate $\hat{C}_{pkT}^*$ for $i^{th}$ subgroup.
   3.3. Repeat 3.1 and 3.2 until the process is declared as out-of-control. Record the number of subgroups until this as a run length.
   3.4. Repeat the above steps 10,000 times to obtain $1/ARL_1$, and then $P = 1/ARL_1$. 

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4. Data envelopment analysis

The design of a control chart requires the specification of three decision variables, i.e. \( n \), \( h \) and \( k \). One of the most powerful methods to solve the mathematical model proposed in this paper (Equation 7) is DEA method. Although DEA is applied for various control charts, it has not been used for the design of the new t-chart yet. In this study, DEA method is used to search the optimal solution(s) in the model.

DEA is a powerful non-parametric approach to evaluate the relative efficiency of a group of decision making units (DMUs) with multiple inputs and outputs. The first DEA approach is known as the CCR model via generalization of the Farrell’s single input, single output efficiency measurement (Charnes et al., 1978). This linear programming formulation can be either input-oriented or output-oriented. Assuming \( n \) DMUs, each with \( m \) inputs and \( q \) outputs, the efficiency of a specific DMU can be obtained by solving the input-oriented CCR model:

\[
\max \quad E_0(S) = \sum_{r=1}^{q} u_r Y_{r0}
\]

\[
\text{s.t.} \quad \sum_{i=1}^{m} v_i X_{i0} = 1
\]

\[
\sum_{r=1}^{q} u_r Y_{rj}(S) - \sum_{i=1}^{m} v_i X_{ij}(S) \leq 0, \quad j = 1, \ldots, n,
\]

\[
u_r \geq 0, \quad r = 1, \ldots, q
\]

\[
v_i \geq 0, \quad i = 1, \ldots, m
\]

where, \( u_r \) is the weight of output \( r \), \( v_i \) is the weight of input \( i \), \( Y_{rj} \) is the value of output \( r \) for \( j \)th DMU, and \( X_{ij} \) is the value of input \( i \) for \( j \)th DMU. The performance of each DMU measured, is relative to the remaining DMUs. A DMU is relatively inefficient if \( E_0^* < 1 \) and relatively efficient, strictly or weakly, if \( E_0^* = 1 \). In designing control charts, DMUs refer to feasible combinations of design parameters. Detailed description of the DEA models is presented in Li et al. (2016).

In the proposed model, the objectives including \( E_L \) and \( ATS_1 \) are considered as inputs because of their minimizing nature, and \( ATS_0 \) is probed as output. The model should be formulated for each DMU to find the set of weights, as decision variables, that maximize the relative efficiency of considered DMU. As a result, at least one of the DMUs will be efficient.

5. Solution procedure

We intend to reach a well-balanced trade-off between the economic and the statistical features. For this reason, Chen and Liao’s algorithm (2004) is employed by some adjustments to optimize the proposed model. Note that in this model the values of objective functions must be computed for each DMU beforehand. The solution algorithm for optimizing the proposed model is described as follows:

**Step 0. Determining the possible combinations of design parameters.** Set various combinations of design parameters by putting bounds on them according to DM’s needs. Due to the discrete optimization nature of the DEA method used in this algorithm, the analyzer can limit the solution space in advance. In this paper, to avoid additional computations related to dominated solutions as our simulation studies showed, we assume \( 3 \leq n \leq 6 \) increases by 1, \( 0.5 \leq h \leq 2 \) increases by 0.5, and \( 2 \leq k \leq 6.5 \) increases by 0.5. Then, the value of each objective function is computed for each DMU.

**Step 1. Determination of feasible combinations.** By using the constraints in equation 7, gather the feasible combinations with the same sample size \( n \) into a set \( Q_n \). Note that DM determines the desired bounds for the constraints of the model in order to limit all objectives for some reasons.

**Step 2. Partial solution selection.** For each set of \( Q_n \), determine the non-dominated solution points that are not dominated in terms of statistical properties and cost in the same set.

**Step 3. Global Pareto solution selection.** Merge all determined solutions from step 2 into a set \( W \) and then select efficient design(s) among the scores calculated by the CCR model.

Although there are some DEA software, in this study, all calculations have been facilitated under a program coded in the MATLAB (version R2013b) environment.
In addition to the DEA, application of evolutionary algorithms, such as: NSGA-II (Li et al., 2016 and Mobin et al., 2015), NSGA-III (Tavana et al., 2016), artificial bee colony algorithm (Vafadarnikjoo et al., 2015), imperialist competitive algorithm (Borghei et al., 2015), the general variable neighborhood search algorithm (Komaki et al., 2015) and other MOEAs presented in Alaei et al. (2016), Fazelzarandi and Kayvanfar (2015), and Kayvanfar et al. (2011), can be considered to solve the multi-objective optimization problem of t control chart design. The optimal solutions, i.e., optimal designs of t control chart obtained by the evolutionary algorithms, can be converted into a manageable size of the optimal solutions by taking three courses of actions: 1) the efficient optimal solutions can be obtained using DEA (Mobin et al., 2015; Li et al., 2016, Tavana et al., 2016); 2) the optimal solutions can be ranked using multiple criteria decision making approaches such as: TOPSIS (Mobin et al., 2014; Salmon et al., 2015); analytical hierarchy process (Mobin et al., 2015), DEMATEL (Vafadarnikjoo et al., 2015), and COPRAS approach (Mobin et al., 2015); and 3) similar optimal solutions can be categorized using the clustering tools such as k-means (Rastegari et al., 2016).

6. Numerical example

In order to illustrate the results of the proposed multi-objective model for the design of new t-chart based on process capability index, the taken random samples of size \( n \) is supposed to follow an exponential distribution with mean \( \theta = 30 \). Moreover, when an assignable cause with the rate of \( \lambda = 0.01 \) occurs, it provides a shift of size \( \delta = 1 \) in the process mean (and so \( \delta_1 = 60 \)). Moreover, \( USL' = 1.805 \) and \( LSL' = 0 \) are considered for the process. The values of other parameters are listed in Table 1. In addition, our constraints for determination of feasible combinations are: \( EL \leq 7 \), \( ATS_0 \geq 86 \) and \( ATS_1 \leq 32 \).

<table>
<thead>
<tr>
<th>Table 1. Input values of parameters</th>
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<tbody>
<tr>
<td>Cost factors</td>
</tr>
<tr>
<td>( M )</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>Time factors</td>
</tr>
<tr>
<td>( g )</td>
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<tr>
<td>0.05</td>
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<tr>
<th>Table 2. Efficient design parameters for the multi-objective model in comparison with pure economic design</th>
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<tbody>
<tr>
<td>Design</td>
</tr>
<tr>
<td>Multi-objective</td>
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<td></td>
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<tr>
<td>Pure Economic</td>
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</tbody>
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Accordingly, Table 2 shows the determined efficient unit in addition to comparison with the pure economic design in which only the expected hourly cost in (6) is minimized (see Duncan’s (1956) model). The selected efficient unit by the proposed model has improved \( ATS_0 \) about 213.9% and \( ATS_1 \) about 18.8%, respectively. However, \( E_L \) is increased about 57.5%. Despite the increase in cost, statistical performance is improved substantially using the proposed multi-objective economic-statistical model. Moreover, in our designs, all the objectives are in the desired limits, while, the statistical objectives of pure economic design are not satisfactory. In terms of efficiency values, our efficient design show significant difference against the pure economic design. This can totally confirm the improved performance of the Multi-Objective Economic-Statistical design and reveal the insufficiency of the pure economic design in such space.

7. Conclusion

With the purpose of optimally determination of design parameters, we proposed a multi-objective economic-statistical model for a new t-chart based on the process capability index. The data envelopment analysis (DEA) approach was employed as solution procedure to specify efficient design parameters. Through a numerical example, the algorithm procedure was investigated in addition to comparison with pure economic design. According to the results, multi-objective economic-statistical design showed statistically improved performance compared to the pure
economic design for the new t-chart based on the process capability index. Using the proposed model in occurrence of multiple assignable causes can be considered as future researches. In addition, models including assignable causes with random shifts and for variable sample sizes in addressing various control charts worth to be investigated.

References


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Biography

Samrad Jafarian Namin is currently a Ph.D. student in Industrial Engineering at Yazd University, Iran. He holds a master degree in Industrial Engineering from Islamic Azad University, South Tehran Branch, Iran, and a bachelor degree from Mazandaran University of Science and Technology, Iran. His fields of interest include Statistical Quality Control (SQC), Time Series Analysis, and Multi Criteria Decision Making (MCDM).

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