$$Max\ \Pi_1(Q,p,S_v,R,w) = kp^{-(\alpha-1)} + \frac{kh_b}{z}Qp^{-\alpha} - \left(\frac{h_v}{2P}+\frac{h_b}{z}\right)kQp^{-\alpha}R^{-1} - krQ^{-\beta}p^{-\alpha}R^{-1} - dkp^{-\alpha}R^{-1}$$
$$- kQ^{-1}p^{-\alpha}S_vR^{-1} - k(S_b+F)Q^{-1}p^{-\alpha}R^{-1} - lkQ^{-1}p^{-\alpha}S_v^{-y}R^{x-1} - \frac{h_b}{2}QR + kwp^{-\alpha} - \frac{kwp^{-\alpha}}{R} \quad (5)$$

subject to, $R/v \le 1$ where, $Q, p, S_v, R, w > 0$.
The problem (5) can be rewritten as

$$Max\ \Pi_1^{'}(Q,p,S_v,R,w) = -Max\ \Pi_1(Q,p,S_v,R,w)$$
$$= -kp^{-(\alpha-1)} - \frac{kh_b}{z}Qp^{-\alpha} + \left(\frac{h_v}{2P}+\frac{h_b}{z}\right)kQp^{-\alpha}R^{-1} + krQ^{-\beta}p^{-\alpha}R^{-1} + dkp^{-\alpha}R^{-1} \quad (6)$$
$$+ kQ^{-1}p^{-\alpha}S_vR^{-1} + k(S_b+F)Q^{-1}p^{-\alpha}R^{-1} + lkQ^{-1}p^{-\alpha}S_v^{-y}R^{x-1} + \frac{h_b}{2}QR - kwp^{-\alpha} + \frac{kwp^{-\alpha}}{R}$$

subject to, $R/v \le 1$, where, $Q, p, S_v, R, w > 0$.
The dual of (6) is given by

$$Max\ \ F(W) = \left(\frac{k}{w_1}\right)^{-w_1}\left(\frac{kh_b}{zw_2}\right)^{-w_2}\left(\frac{k(h_v/(2P)+h_b/z)}{w_3}\right)^{w_3}\left(\frac{kr}{w_4}\right)^{w_4}\left(\frac{dk}{w_5}\right)^{w_5}\left(\frac{k}{w_6}\right)^{w_6}\left(\frac{k(S_b+F)}{w_7}\right)^{w_7}\left(\frac{lk}{w_8}\right)^{w_8}\left(\frac{h_b}{2w_9}\right)^{w_9}\left(\frac{k}{w_{10}}\right)^{-w_{10}}$$
$$\times\left(\frac{k}{w_{11}}\right)^{w_{11}}\left(\frac{1}{v}\right)^{w_{12}} \quad (7)$$

Subject to the normality and orthogonality conditions

$$-w_1 - w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 + w_9 - w_{10} + w_{11} = 1$$
$$-w_2 + w_3 - \beta w_4 - w_6 - w_7 - w_8 + w_9 = 0$$
$$(\alpha-1)w_1 + \alpha w_2 - \alpha w_3 - \alpha w_4 - \alpha w_5 - \alpha w_6 - \alpha w_7 - \alpha w_8 + \alpha w_{10} - \alpha w_{11} = 0 \quad (8)$$
$$w_6 - yw_8 = 0$$
$$-w_3 - w_4 - w_5 - w_6 - w_7 + (x-1)w_8 + w_9 - w_{11} + w_{12} = 0$$
$$-w_{10} + w_{11} = 0$$

The non-negativity conditions are

$$-w_1 - w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 + w_9 - w_{10} + w_{11} \ge 0$$
$$w_j \ge 0,\ \forall\ j = 1,2,3,....12$$

Solving the above five equations (8) in terms of seven independent variables (DD=7) we get the optimal values of the primal variables. The system of dual equations in (8) are six linear equations in 13 variables which can be written as $A^TW = B$, where

$$A^T = \begin{pmatrix} -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -\beta & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ \alpha-1 & \alpha & -\alpha & -\alpha & -\alpha & -\alpha & -\alpha & -\alpha & 0 & \alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -y & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & -1 & -1 & x-1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$W = [w_1\ w_2\ \ w_3\ w_4\ w_5\ w_6\ w_7\ w_8\ w_9\ w_{10}\ w_{11}\ w_{12}]^T$

We can write six equations in six unknowns in terms of 12-6 = 6 independent variables. Without loss of generality, we consider w_1, w_2, w_5, w_6, w_7, w_{10} as dependent variables and remaining variables w_3, w_4, w_8, w_9, w_{11}, w_{12} as independent variables. We consider $W_1 = [w_1\ w_2\ \ w_5\ \ w_6\ w_7\ w_{10}]^T$ and $W_2 = [w_3\ \ w_4\ \ w_8\ w_9\ w_{11}\ \ w_{12}]^T$ and A_1 = Coefficient matrix (non-singular) of six dependent variables.

$$A_1 = \begin{bmatrix} -1 & -1 & 1 & 1 & 1 & -1 \\ 0 & -1 & 0 & -1 & -1 & 0 \\ \alpha-1 & \alpha & -\alpha & -\alpha & -\alpha & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

A_2 = Coefficient matrix (non-singular) of six independent variables.

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -\beta & -1 & 1 & 0 & 0 \\ -\alpha & -\alpha & -\alpha & 0 & -\alpha & 0 \\ 0 & 0 & -y & 0 & 0 & 0 \\ -1 & 1 & x-1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then the relation $A^T W = B$ is transformed into $A_1W_1+A_2W_2 = B$ i.e., $W_1 = A_1^{-1}B - A_1^{-1}A_2W_2$. Considering suitable values of w_3, w_4, w_8, w_9, w_{11}, w_{12} satisfying normality, orthogonality and non-negativity conditions as stated in (8) we can find the optimal values of w_1, w_2, w_5, w_6, w_7, w_{10}. Let these optimal values be w_1^*, w_2^*, w_5^*, w_6^*, w_7^*, w_{10}^*, respectively. Now the optimal values of *p, Q, R, S_v,* and *w* are found from the relations:

$$\frac{kp^{-\alpha-1}}{w_1} = \frac{kh_bQp^{-\alpha}}{w_2 z} = \frac{dkp^{-\alpha}R^{-1}}{w_5} = \frac{kQ^{-\alpha}p^{-\alpha}S_vR^{-1}}{w_6} = \frac{k(S_b+F)Q^{-1}p^{-\alpha}R^{-1}}{w_7} = \frac{kwp^{-\alpha}}{w_{10}}$$

Thus, the optimal solutions of the problem are summarized as follows:

$$Q^* = \frac{k(S_b+F)w_6^*}{dkw_8^*},\ p^* = \frac{h_b w_1^* Q^*}{zw_2^*},\ S_v^* = \frac{(S_b+F)w_7^*}{w_8^*},\ R^* = \frac{dzw_2^*}{h_bQ^*w_6^*},\ w^* = \frac{h_bw_6^*w_{10}^*Q^*}{zw_2^*w_5^*}$$

5.2 Case II: *n* = 2

In this case, the model (4) is transformed into

$$\begin{aligned} Max\ \Pi_2(Q,p,S_v,R,w) &= kp^{-(\alpha-1)} + \frac{kh_b}{z}Qp^{-\alpha} - \frac{h_b}{z}kQp^{-\alpha}R^{-1} + h_vQ/2 - krQ^{-\beta}p^{-\alpha}R^{-1} - dkp^{-\alpha}R^{-1} \\ &- (k/2)Q^{-1}p^{-\alpha}S_vR^{-1} - k(S_b/2+F)Q^{-1}p^{-\alpha}R^{-1} - (lk/2)Q^{-1}p^{-\alpha}S_v^{-y}R^{x-1} - \frac{h_b}{2}QR + kwp^{-\alpha} - \frac{kwp^{-\alpha}}{R} \end{aligned} \tag{9}$$

subject to, $R/v \le 1$ where, $Q, p, S_v, R, w > 0$.
The above problem can be rewritten as

$$\begin{aligned} Max\ \Pi_2'(Q,p,S_v,R,w) &= -Max\ \Pi_2(Q,p,S_v,R,w) \\ &= -kp^{-(\alpha-1)} - \frac{kh_b}{z}Qp^{-\alpha} + \frac{h_b}{z}kQp^{-\alpha}R^{-1} - (h_v/2)Q + krQ^{-\beta}p^{-\alpha}R^{-1} + dkp^{-\alpha}R^{-1} \\ &+ (k/2)Q^{-1}p^{-\alpha}S_vR^{-1} + k(S_b/2+F)Q^{-1}p^{-\alpha}R^{-1} + (lk/2)Q^{-1}p^{-\alpha}S_v^{-y}R^{x-1} + \frac{h_b}{2}QR - kwp^{-\alpha} + \frac{kwp^{-\alpha}}{R} \end{aligned} \tag{10}$$

subject to, $R/v \le 1$ where, $Q, p, S_v, R, w > 0$.
The above problem is a signomial Geometric Programming problem with DD=7. The dual of (10) is given by

$$\begin{aligned} Max\ \ F(\delta) &= \left(\frac{k}{\delta_1}\right)^{-\delta_1}\left(\frac{kh_b}{z\delta_2}\right)^{-\delta_2}\left(\frac{kh_b/z}{\delta_3}\right)^{\delta_3}\left(\frac{hv}{2\delta_4}\right)^{\delta_4}\left(\frac{kr}{\delta_5}\right)^{\delta_5}\left(\frac{dk}{\delta_6}\right)^{\delta_6}\left(\frac{k}{2\delta_7}\right)^{\delta_7}\left(\frac{k(S_b/2+F)}{\delta_8}\right)^{\delta_8}\left(\frac{lk}{2\delta_9}\right)^{\delta_9}\left(\frac{h_b}{2\delta_{10}}\right)^{\delta_{10}}\left(\frac{k}{\delta_{11}}\right)^{-\delta_{11}} \\ &\times\left(\frac{k}{\delta_{12}}\right)^{\delta_{12}}\left(\frac{1}{v}\right)^{\delta_{13}} \end{aligned} \tag{11}$$

Subject to the normality and orthogonality conditions

$$\begin{aligned} &-\delta_1 - \delta_2 + \delta_3 - \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8 + \delta_9 + \delta_{10} - \delta_{11} + \delta_{12} = 1 \\ &-\delta_2 + \delta_3 - \delta_4 - \beta\delta_5 - \delta_7 - \delta_8 - \delta_9 + \delta_{10} = 0 \\ &(\alpha-1)\delta_1 + \alpha\delta_2 - \alpha\delta_3 - \alpha\delta_5 - \alpha\delta_6 - \alpha\delta_7 - \alpha\delta_8 - \alpha\delta_9 + \alpha\delta_{11} - \alpha\delta_{12} = 0 \\ &\delta_7 - y\delta_9 = 0 \\ &-\delta_3 - \delta_5 - \delta_6 - \delta_7 - \delta_8 + (x-1)\delta_9 + \delta_{10} - \delta_{12} + \delta_{13} = 0 \\ &-\delta_{11} + \delta_{12} = 0 \end{aligned} \tag{12}$$

The non-negativity conditions are

$$-\delta_1 - \delta_2 + \delta_3 - \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8 + \delta_9 + \delta_{10} - \delta_{11} + \delta_{12} \ge 0$$
$$\delta_j \ge 0,\ \forall\ j = 1,2,3,....13$$

We consider dependent variables $\delta_2,\delta_3,\delta_4,\delta_7,\delta_8,\delta_{11}$ in terms of seven independent variables $\delta_1,\delta_5,\delta_6,\delta_9,\delta_{10},\delta_{12},\delta_{13}$ as satisfying normality, orthogonality and non-negativity conditions stated in (12). Let $\delta_2^*,\delta_3^*,\delta_4^*,\delta_7^*,\delta_8^*,\delta_{11}^*$ be respective optimal values of dependent variables for suitably chosen $\delta_1,\delta_5,\delta_6,\delta_9,\delta_{10},\delta_{12},\delta_{13}$. One of the optimal solutions of the problem is found as

$$Q^* = \sqrt{\frac{z(S_b/2+F)\delta_3^*}{h_b\delta_8^*}},\ p^* = \left(\frac{2kh_b\delta_4^*}{zh_v\delta_2^*}\right)^{1/\alpha},\ S_v^* = \frac{(S_b/2+F)\delta_7^*}{\delta_8^*},\ R^* = \frac{\delta_2^*}{\delta_3^*},\ w^* = \frac{h_b\delta_{11}^*Q^*}{z\delta_2^*}$$

from the relations:

$$\frac{kh_bQp^{-\alpha}}{z\delta_2} = \frac{kh_bQp^{-\alpha}R^{-1}}{z\delta_3} = \frac{h_vQ}{2\delta_4} = \frac{kQ^{-\alpha}p^{-\alpha}S_vR^{-1}}{2\delta_7} = \frac{k(S_b/2+F)Q^{-1}p^{-\alpha}R^{-1}}{\delta_8} = \frac{kwp^{-\alpha}}{\delta_{11}}$$

5.3 Case III: $n \geq 3$

We first consider the subcase(i): $\frac{(n-2)h_v}{2P} > \frac{h_b}{z}$ for $n = n^* \geq 3$. In this subcase, the model (4) is a signomial Geometric Programming problem with degree of difficulties 13-5-1=7. Its dual is given by

$$Max\ \ F(\gamma) = \left(\frac{k}{\gamma_1}\right)^{-\gamma_1}\left(\frac{kh_b}{z\gamma_2}\right)^{-\gamma_2}\left(\frac{k(n^*-2)h_v/(2P)-h_b/z}{\gamma_3}\right)^{-\gamma_3}\left(\frac{hv(n^*-1)/2}{\gamma_4}\right)^{\gamma_4}\left(\frac{kr}{\gamma_5}\right)^{\gamma_5}\left(\frac{dk}{\gamma_6}\right)^{\gamma_6}\left(\frac{k}{n^*\gamma_7}\right)^{\gamma_7}\left(\frac{k(S_b/n^*+F)}{\gamma_8}\right)^{\gamma_8}\left(\frac{lk}{n^*\gamma_9}\right)^{\gamma_9}$$
$$\times\left(\frac{h_b}{2\gamma_{10}}\right)^{\gamma_{10}}\left(\frac{k}{\gamma_{11}}\right)^{-\gamma_{11}}\left(\frac{k}{\gamma_{12}}\right)^{\gamma_{12}}\left(\frac{1}{v}\right)^{\gamma_{13}}$$

subject to the normality and orthogonality conditions

$$\begin{aligned}
&-\gamma_1-\gamma_2-\gamma_3+\gamma_4+\gamma_5+\gamma_6+\gamma_7+\gamma_8+\gamma_9+\gamma_{10}+\gamma_{11}-\gamma_{12}=1\\
&-\gamma_2-\gamma_3+\gamma_4-\beta\gamma_5-\gamma_7-\gamma_8-\gamma_9+\gamma_{10}=0\\
&(\alpha-1)\gamma_1+\alpha\gamma_2+\alpha\gamma_3-\alpha\gamma_5-\alpha\gamma_6-\alpha\gamma_7-\alpha\gamma_8-\alpha\gamma_9+\alpha\gamma_{11}-\alpha\gamma_{12}=0\\
&\gamma_7-y\gamma_9=0\\
&-\gamma_3-\gamma_5-\gamma_6-\gamma_7-\gamma_8+(x-1)\gamma_9+\gamma_{10}-\gamma_{12}+\gamma_{13}=0\\
&-\gamma_{11}+\gamma_{12}=0
\end{aligned} \tag{13}$$

The non-negativity conditions are

$$-\gamma_1-\gamma_2-\gamma_3+\gamma_4+\gamma_5+\gamma_6+\gamma_7+\gamma_8+\gamma_9+\gamma_{10}-\gamma_{11}+\gamma_{12}\geq 0$$
$$\gamma_j \geq 0,\ \forall\ j = 1,2,3,....12$$

where the primal problem is

$$Max\ \Pi_3'(Q,p,S_v,R,w) = -Max\ \Pi_3(Q,p,S_v,R,w)$$
$$= -kp^{-(\alpha-1)} - \frac{kh_b}{z}Qp^{-\alpha} + \left(\frac{(n^*-2)h_v}{2P} - \frac{h_b}{z}\right)kQp^{-\alpha}R^{-1} + \frac{(n^*-1)}{2}h_vQ + krQ^{-\beta}p^{-\alpha}R^{-1} + dkp^{-\alpha}R^{-1}$$
$$+(k/n^*)Q^{-1}p^{-\alpha}S_vR^{-1} + k(S_b/n^*+F)Q^{-1}p^{-\alpha}R^{-1} + (lk/n^*)Q^{-1}p^{-\alpha}S_v^{-y}R^{x-1} + \frac{h_b}{2}QR - kwp^{-\alpha} + \frac{kwp^{-\alpha}}{R}$$

subject to, $R/v \leq 1$ where, $Q, p, S_v, R, w > 0$.

In this case, without loss of generality, we consider six dependent variables $\gamma_2,\gamma_3,\gamma_4,\gamma_7,\gamma_8,\gamma_{11}$ in terms of seven independent variables $\gamma_1,\gamma_5,\gamma_6,\gamma_9,\gamma_{10},\gamma_{12},\gamma_{13}$ satisfying normality, orthogonality and non-negativity conditions as stated in (13). Let $\gamma_2^*,\gamma_3^*,\gamma_4^*,\gamma_7^*,\gamma_8^*,\gamma_{11}^*$ be the respective optimal values of dependent variables for suitable choice of $\gamma_1,\gamma_5,\gamma_6,\gamma_9,\gamma_{10},\gamma_{12},\gamma_{13}$.

Then from the relations

$$\frac{kh_bQp^{-\alpha}}{z\gamma_2} = \frac{(\frac{(n^*-2)h_v}{2P} - \frac{h_b}{z})kQp^{-\alpha}R^{-1}}{\gamma_3} = \frac{(n^*-1)h_vQ}{2\delta_4} = \frac{kQ^{-1}p^{-\alpha}S_vR^{-1}}{n^*\gamma_7} = \frac{k(S_b/n^*+F)Q^{-1}p^{-\alpha}R^{-1}}{\gamma_8} = \frac{kwp^{-\alpha}}{\gamma_{11}}$$

one of the optimal solutions is found as

$$Q^* = \sqrt{\frac{S_v^*\gamma_3^*}{n^*\gamma_7^*((n^*-2)h_v/(2P)-h_b/z)}},\ p^* = \left(\frac{2kh_b\gamma_4^*}{(n^*-1)zh_v\gamma_2^*}\right)^{1/\alpha},\ S_v^* = \frac{(S_b+n^*F)\gamma_7^*}{\gamma_8^*},\ R^* = \frac{z\gamma_2^*((n^*-2)h_v/(2P)-h_b/z)}{h_b\gamma_3^*},\ w^* = \frac{h_b\gamma_{11}^*Q^*}{z\gamma_2^*}$$

In the subcase(ii): $\frac{(n-2)h_v}{2P} < \frac{h_b}{z}$ for $n = n^* \geq 3$, proceeding similarly as before, we can find the optimal values of Q, p, S_v, R, w as follows:

$$Q^* = \sqrt{\frac{S_v^* \mu_3^*}{n^* \mu_7^* (-(n^*-2)h_v/(2P) + h_v/z)}},\ p^* = \left(\frac{2kh_b\mu_4^*}{(n^*-1)zh_v\mu_2^*}\right)^{1/\alpha},\ S_v^* = \frac{(S_b + n^*F)\mu_7^*}{\mu_8^*},\ R^* = \frac{z\mu_2^*(-(n^*-2)h_v/(2P) + h_b/z)}{h_b\mu_3^*},\ w^* = \frac{h_b\mu_{11}^* Q^*}{z\mu_2^*}$$

6. Numerical Example

In this section, we illustrate the developed models through a numerical example. We assume the parameter-values as $\alpha = 2.5$, $\beta = 0.02$, $r = 5$, $k = 10^6$, $l = 80000$, $x = 1.2$, $y = 4.8$, $P = 160000$, $S_b = 30$, $h_v = 2$, $h_b = 5$, $F = 25$, $z = 175200$, $d = 0.5$, $W = 6$, $v = 0.99$ in appropriate units. For this chosen data, we see that integrated profit is maximum for $n^* = 7$, $Q^* = 155.36$ units, $p^* = 8.69$, $w^* = 3.09$ (each defective item is sold at (6-3.09) = 2.91), $S_v = 9.1794$, and $R^* = 0.99$. We observe that annual demand of the buyer is 4492.37 units. So, the vendor has to produce 4537.75 units in a year. Here, unit production cost per item reduces to 4.52 and hence the total production cost per year is 20510.85. In a year, the vendor earns 26954.22 from selling products whereas his/her total expenses is 21607.41 and hence the vendor's total annual profit reduces to 5346 .81. On the other hand, the buyer earns 39037.80 from selling products whereas his/her total expense is 30463.19 and hence the buyer's total annual profit reduces to 8574.61. As a result, annual profit of the supply chain becomes 13921.42.

6.1 The case of $\beta = 0$ (unit production cost does not depend on lot size)

Here we assume that unit production cost is fixed and it is 5. We observe that annual demand of the buyer decreases in this case and it is 3451.85 units. So the vendor has to produce 3486 .72 units in a year and hence total production cost per year is 1743.58. Since demand is less, the buyer wants to receive less items from the vendor in each replenishment. The vendor's set-up cost remains unchanged but the defective items are sold with more discount. The buyer compensates this loss by selling the good items with higher price. In this case, sales revenues of the vendor and buyer are 20711.10 (23% decrease) and 33329.68 (14.62% decrease), respectively, and the profits of the vendor, the buyer and the whole supply chain are 2306.91 (56.85% decrease), 9634.52 (12.36% increase) and 11941.44 (14.22% increase), respectively. It is clear that the model with lot size dependent unit production cost is more profitable than fixed unit production cost.

6.2 The case of $R = 1$ (all items produced by the vendor are good)

In real market, increase in reliability of a product increases the demand of the product. We observe that annual demand of the buyer is 4719.55 units, which is larger than that of our proposed model. Here the vendor produces no defective item and hence, all the products meet to customer demand. So the vendor need not produce extra units. Therefore, the vendor has to produce 4719.55 units in a year. Here unit production cost reduces to 4.43 and hence the total production cost per year becomes 20905.88. Since customers' demand is high, so the buyer wants to receive more items in each replenishment from the vendor. The vendor or the buyer has to pay no extra cost to the customer as there is no defective items. As the vendor has to produce more good items, he/she has to set-up super-structural machine and use new/ultramodern technique for which vendor's set-up cost increases. Here the sales revenues of the vendor and the buyer are 28317.33(5% increase) and 40210.60 (3% increase), respectively, and the profits of the vendor, the buyer and the integrated supply chain are 7185.86 (34.4% increase), 7858.41(8.35% increase) and 15044.27 (8.07% increase), respectively. It is clear from the above results that if a product is highly reliable then profit of the supply chain will be more.

Variable	Our proposed model	Different cases	
		$\beta = 0$	$R = 1$
n	7	9	1
p	8.6898	9.6556(11.11%)	8.5155(-2.01%)
Q	155.3596	99.3046(-36.08%)	426.7090(174.66%)
S_v	9.1794	9.1790(0.0%)	19.2026(+109.19%)
w	3.0908	4.2484(37.45%)	0.0000(-100.00%)

From Table 1, we can conclude that customers' demand and lot size in each replenishment are proportional whereas lot size and selling price are complementary; if customers' demand is high then the buyer's selling price is low and vice-versa.

6.3 Comparison of the optimal results with those of Huang's (2004) model

To compare the optimal results with those of Huang's (2004) model, we take $\alpha = 0$, $k = 50000$, $l = 0$, $x = 0$, $y = 0$. Huang considered the expected number of defective items as 0.02 and hence we consider $R = 0.98$. We further consider $S_b = 100$, $S_v = 300$, $w = 30$. All the remaining parameters P, h_v, h_b, F, z (it is x in Huang's model), and d are kept unchanged to match with Huang's model. Comparing these two models, we have the following results: For given n, the average total cost of the integrated system is a function of Q only and this function is a convex function in $Q > 0$. We determine the unique value of Q as

$$Q^* = \sqrt{\frac{2kPz(nF + S_b + S_v)}{np^{\alpha} PRz\{(n-1)h_v + h_b R\} + nk\{2h_b P(R-1) - (n-2)zh_v\}}} \quad (14)$$

and hence the average total cost of the integrated system becomes

$$\frac{p^{-\alpha}}{nR}\left(\sqrt{\frac{2k(nF + S_b + S_v)(np^{\alpha} PRz\{(n-1)h_v + h_b R\} + nk\{2h_b P(R-1) - (n-2)zh_v\})}{zP}} - nk\{(R-1)w - d\}\right) \quad (15)$$

With the chosen data set, the optimal results are obtained as n^*=7 and Q^*=780.21 and the corresponding average total cost of the supply chain is 66865.60. Huang (2004) obtained this cost as 67082.60. Hence our model outperforms Huang's (2004) model in terms of average total cost of the whole supply chain.

Now, we consider the model from buyer's perspective. The buyer's annual cost is given by

$$\frac{dkp^{-\alpha}}{R} + \frac{Fkp^{-\alpha}}{QR} + \frac{kp^{-\alpha} S_b}{nQR} + \frac{QRh_b}{2} + \frac{kp^{-\alpha} Q(1-R)}{Rz} \quad (16)$$

which can be shown to be a convex function in $Q(>0)$. Therefore, there exists a unique value

$$Q^{**} = \sqrt{\frac{2k(nF + S_b)z}{nh_b\{2k(R-1) + p^{\alpha} R^2 z\}}} \quad (17)$$

With the chosen data set, the optimal value of Q is obtained as Q^{**}= 1603.90. Substituting Q^{**}= 1603.90 and n = 1 into the buyer's annual cost and the vendor's annual cost yield the values 33462.7 and 40666.8, respectively (Huang obtained these values as 33467.17 and 40881.64, respectively). Hence, there is a cost reduction of (33462.7 + 40666.8 − 66865.6) =197263 .9 (in Huang's model, this reduction is 7265 .21) in the integrated approach, which is as impressive as Huang's model.

7. Conclusions and Future Research Directions

In this paper, we consider a supply chain model with a single-vendor and a single-buyer where the market demand depends on selling price of the product. All items produced by the vendor may not be perfect. The buyer screens all the products collected by him/her from the vendor. We optimize the profit of the supply chain with respect to the number of shipments, shipment size, buyer's selling price, the vendor's setup cost and the discount cost of the defective products. We apply signomial Geometrical Programming technique to optimize the profit. We analyze the impacts of sales revenue, different costs and profits of the vendor, the buyer and the whole supply chain on the price elasticity to demand, lot size elasticity to unit production cost, unit screening cost, scaling constant for demand, scaling constant for unit production cost and reliability of the product.

From the numerical study, we observe that lot size dependent unit production cost is more sensitive than constant unit production cost. If lot size increases then unit production cost decreases, and hence profit of the vendor increases. In real situations, it is impossible to produce all perfect items. The profits of the vendor, the buyer and the whole supply chain increase if the vendor produce more reliable items. Also, we investigate that it is necessary for the vendor to produce items not more than certain percentage (in our numerical example, 14%) of defective items.

The scaling constant for the demand also acts an important role to increase the profit of the buyer, the vendor and the whole supply chain. The price elasticity to demand must maintain the interval which lies between 1 and 3. We construct our model similar to any other model, based on a set of assumptions. We have assumed market demand as deterministic, which has limited applications in the real world. So, one can consider stochastic demand instead of deterministic demand to extend this model for further research. We have assumed two-layer supply chain model with a single vendor and a single buyer. Further research can extend the model by considering multi-layer, multi-buyer and/or multi-vendor supply chain. The vendor may impose 'terms and conditions' when selling the defective items with discount price.

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