Review Methods to Solve Fractional Black-Scholes

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Abstract

The Black-Scholes model became popular and almost universally accepted by option traders to estimate and assess option prices over time. Black-Scholes model involves several assumptions that must be fulfilled, so sometimes the Black-Scholes model is difficult to apply to real life under certain conditions because of these assumptions. Therefore, several models were developed to approach these assumptions, one of them is fractional Black-Scholes model. The model used in this paper is time-fractional Black-Scholes model. This paper begins with a brief description of the history of the Black-Scholes model then followed by an explanation of time-fractional Black-Scholes obtained from ordinary Black-Scholes models. The main points highlighted in this paper are the methods of solving the fractional Black-Scholes model. This paper aims to review the methods that have been successfully used in previous studies. Like Laplace Homotopy Perturbation Method (LHPM), fractional variational iteration method, Block-Pulse operational matrix method, Homo-Separation of Variables method, and several other methods. In reviewing these methods, this paper presents summary of the contexts that are worked out using these methods, conclusions and the efficiency of the methods, so that differences among the methods can be seen clearly.

Keywords : Option, Fractional, Black-Scholes Model, Methods.

1. Introduction

Determining option prices is a major problem in quantitative finance. These problems are theoretical and practical problems because of the use of options in the financial industry today. In the option pricing theory, the Black-Scholes equation is one of the most effective models for option pricing. Black-Scholes is modeled with stock price movements as a stochastic process by adding a number of assumptions related to option markets. These assumptions are the chance of arbitration and tax does not exist, interest is risk free and the volatility is constant (Hull and White, 1987). In fact, these assumptions are often difficult to adapt to existing conditions. Therefore, to approach these assumptions several models are developed, one of which is the Black-Scholes fractional model (Chen, 2014). Then, there are many methods used to determine the solution of the fractional model, both analytically and numerically. In this study, a number of methods for solving the Black-Scholes fractional model are summarized, in terms of the assumptions involved in the method, the efficiency of these methods, and conclusions obtained after using these methods as a method of completing the Black-Scholes fractional model. Before that, first also told about the process of the beginning of the emergence of this famous Black-Scholes model. Accompanied by a description of the formation of the Black-Scholes model mathematically.

History of Black-Scholes model

Financial derivatives are financial assets with pay-off prices depending on the underlying assets, such as stocks, indices, interest rates, foreign exchange rates, etc. Option is derivative securities that give the owner the right to buy (call option) or sell (put option) the underlying asset at the specified price (strike price) at or before a certain time. The financial market model using a stochastic process has begun in these decades. In the 1900s, Bachelier modeled the stock price as a Brownian movement with drift, but in this model there was a clear lack, namely stock prices could be negative (Chen, 2014). Based on this model, in 1973 Fisher Black & Myron Scholes modeled that European option prices on a stock whose price follows Brownian geometry with drift and constant volatility, as a second-order partial differential equation, which became known as the Black-Scholes equation (Black & Scholes, 1973). The Black-Scholes analysis assumes that stock prices behave as just demonstrated. Further assumptions of the analysis are as follows (Hackmann, 2009) :

1. Investors are permitted to short sell stock. That is, it is permitted to borrow stock with the intent to sell and use the proceeds for other investment. Selling an asset without owning it is known as having a short position in that asset, while buying and holding an asset is known as having a long position.
2. There are no transaction costs or taxes.
3. The underlying stock does not pay dividends.
4. There are no arbitrage opportunities. That is, it is not possible to make risk free investments with a return greater than the risk free rate.
5. Stock can be purchased in any real quantity. That is, it is possible to buy π shares of stock or 1/100 shares of stock.
6. The risk free rate of interest r is constant.

The history of options markets goes back to the middle ages, when the futures were created in order to meet the need of merchants and farmers. Consider the position of a farmer in March who will harvest in June. Now, he is uncertain about the price of grains. The option market was developed to manage these kinds of risks. Later on, these trades were formalized through the trading boards, when the Chicago Board of Trade was established in 1848 in order to bring the merchants and farmers onto a single platform (Al-Saedi and Tularam, 2018). It was then that academicians and financial researchers started to focus on the valuation methods, such as the Black-Scholes equation to model the prices of the options. Despite some estimation issues, as pointed out in Harun and Hafizah (2015), the Black-Scholes model has received considerable attention over past two decades-especially in underlying probability attributes of a European call option when written on a non-dividend stock. The Black-Scholes model estimates the probability of a European call option, which is frequently used in the investment decisions (Black and Scholes, 1973). The focus was not on American options since the European one was much easier to deal with at the time. An acknowledgement of the importance of the Black-Scholes model came in 1997, when Myron Scholes and Robert Merton were awarded the Nobel Prize in 1997. Fischer Black passed away on 30th August 1995; otherwise, he would undoubtedly also have been one of the recipients of the Nobel Prize (Al-Saedi and Tularam, 2018). This model has now become one of the most important applications of Ito calculus in financial engineering. In this context, Wilmott et al. (1995) pointed out that the Black-Scholes or Black-Scholes-Merton model is the basic building block of the financial derivatives theory. Further, this model played a vital role in the growth and success of financial engineering.

However, this Black-scholes model cannot correctly capture the dynamics of the option prices because the empirical data shows that the assumption of log-normal diffusion with constant volatility is not consistent with the market prices. One phenomenon that exists in all stock markets is the volatility skew or smile (Chen, 2014). To improve the performance of the Black-Scholes model, several extensions of the Black-Scholes model were developed, such as the jump diffusion models (Hirsa & Madan, 2004)(Kou, 2002), stochastic volatility models (Hull, 2013), etc. Black - Scholes model has Gaussian shocks which underestimate the probability of an extreme movement in the stock price than that these model suggests. So more realistic models have been proposed to model the movements in the stock price. One such model is called FMLS (Finite Moment Log Stable) model which falls in the class of L´evy models whose process can be written as fractional partial diffusion type equations (Chen, 2004).

The Black-Scholes equation and boundary conditions for an European call option with value $C(S,\tau)$ is

$$\frac{\partial C}{\partial \tau}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0 \tag{1}$$

with $C(0,\tau)=0,\ \ C(S,\tau)\sim S,$ as $S\to\infty,$ and $C(S,T)=\max\{S-E,0\}$

where σ is the volatility of the underlying asset, E is the exercise price, T is the expiry time and r is the risk free interest rate.

Equation (1) is a diffusion equation but each time C is differentiated with respect to S and it is multiplied by S giving non-constant coefficients. So, set

$$S=Ee^x,\quad \tau=T-\frac{t}{(1/2)\sigma^2},\quad C=Ev(x,t)\ ,$$

because $S=Ee^x$, then obtained that $x=\ln\left(\frac{S}{E}\right)$. $\tau=T-\frac{t}{(1/2)\sigma^2}$, then obtained that $t=\frac{\sigma^2}{2}\left(T-\tau\right)$. By using chain derivative then obtained

$$\frac{\partial C}{\partial \tau}=\frac{\partial Ev(x,t)}{\partial \tau}=E\frac{\partial v}{\partial t}.\frac{\partial t}{\partial \tau}=-E\frac{\sigma^2}{2}.\frac{\partial v}{\partial t} \tag{2}$$

$$\frac{\partial C}{\partial S}=\frac{\partial Ev(x,t)}{\partial S}=E\frac{\partial v}{\partial x}.\frac{\partial x}{\partial S}=E\frac{1}{S}.\frac{\partial v}{\partial x} \tag{3}$$

$$\frac{\partial^2 C}{\partial S^2}=\frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right)=\frac{\partial}{\partial S}\left(E.\frac{1}{S}.\frac{\partial v}{\partial x}\right)=-\frac{E}{S^2}.\frac{\partial v}{\partial x}+\frac{E}{S^2}.\frac{\partial^2 v}{\partial x^2} \tag{4}$$

by substituting equation (2), (3), and (4) to equation (1), then equation (1) become

$$-E\frac{\sigma^2}{2}.\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 S^2\left(-\frac{E}{S^2}.\frac{\partial v}{\partial x}+\frac{E}{S^2}.\frac{\partial^2 v}{\partial x^2}\right)+rSE\frac{1}{S}.\frac{\partial v}{\partial x}-rEv=0, \tag{5}$$

simplify equation (5) , after that devided by E, so equation (5) become

$$-\frac{\sigma^2}{2}.\frac{\partial v}{\partial t}-\frac{\sigma^2}{2}.\frac{\partial v}{\partial x}+\frac{\sigma^2}{2}.\frac{\partial^2 v}{\partial x^2}+r.\frac{\partial v}{\partial x}-rv=0, \tag{6}$$

then, devided equation (6) with $-\frac{2}{\sigma^2}$, it will be obtained

$$\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}-\frac{\partial^2 v}{\partial x^2}-\frac{2r}{\sigma^2}.\frac{\partial v}{\partial x}+\frac{2r}{\sigma^2}.v=0. \tag{7}$$

Suppose $k=\frac{2r}{\sigma^2}$, then equation (7) become

$$\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}-\frac{\partial^2 v}{\partial x^2}-k\frac{\partial v}{\partial x}+kv=0,$$

so that equation (1) derived by that way above turn out

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+(k-1)\frac{\partial v}{\partial x}-kv, \tag{8}$$

In the form of fractional order, equation (8) become

$$\frac{\partial^\alpha v}{\partial t^\alpha}=\frac{\partial^2 v}{\partial x^2}+(k-1)\frac{\partial v}{\partial x}-kv,\qquad 0<\alpha\le 1\ . \tag{9}$$

Equation (9) is known as time-fractional Black-Scholes.

2. Methods to solve fractional Black-Scholes model

2.1 Laplace Homotopy Perturbation Method

Kumar et al (2012) was determining the fractional Black-Scholes using Laplace homotopy perturbation method (LHPM). Laplace homotopy perturbation method, which is combined form of the Laplace transform and the homotopy perturbation method, is employed to obtain a quick and accurate solution to the fractional Black Scholes equation with boundary condition for a European option pricing problem. The Black-Scholes formula is used as a model for valuing European or American call and put options on a non-dividend paying stock. The proposed scheme finds the solutions without any discretization or restrictive assumptions and is free from round-off errors and therefore, reduces the numerical computations to a great extent. The analytical solution of the fractional Black Scholes equation is calculated in the form of a convergent power series with easily computable components.

The LHPM basically illustrates how the Laplace transform can be used to approximate the solutions of the linear and nonlinear differential equations by manipulating the homotopy perturbation method which was first introduced and applied by He (He, 2005)(He, 2006). The LHPM method is very well suited to physical problems since it does not require unnecessary linearization, perturbation and other restrictive methods and assumptions which may change the problem being solved, sometimes seriously (Kumar, 2012). The main advantage of this method is to overcome the deficiency that is mainly caused by unsatisfied conditions. Thus, it is said that the LHPM methodology is very powerful and efficient in finding approximate solutions as well as numerical solutions.

The equation that used in this research is equation (9) and it is obtained that the solution $v(x,t)$ of the problem given by

$$v(x,t)=\lim_{p\to 1}\sum_{n=0}^{\infty} p^i v_i(x,t)=\max\left(e^x-1,0\right)E_\alpha\left(-kt^\alpha\right)+\max\left(e^x,0\right)\left(1-E_\alpha\left(-kt^\alpha\right)\right) , \tag{10}$$

where $E_\alpha(z)$ is Mittag-Leffler function in one parameter. The solution above is closed form solution of the fractional Black-Scholes equation (9).

2.2 Fractional Variational Iteration Method

Elbeleze et al (2013) use the fractional variational iteration method (FVIM) with modified Riemann-Liouville derivative to solve some equations in fluid mechanics and in financial models. The fractional derivatives are described in Riemann-Liouville sense. The variational iteration method is one of approaches to provide an analytical approximation solution to linear and nonlinear problems (Ganji and Sadigi, 2007) (Ozer, 2007). The fractional variational iteration method with Riemann-Liouville derivative was proposed by Wu and Lee (2010) and applied to solve time fractional and space fractional diffusion equations. They considered the following fractional differential equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\partial}=R\left[x\right]u(x,t)+q(x,t) \ , \qquad 0<\alpha\le 1, x\in\mathbf{R}, t>0,$$

with initial condition $u(x,0)=f(x)$.

By using this method, it is obtained the general form for iteration formula, that is

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)-\frac{\delta}{\Gamma(\alpha+1)}\times\int_0^t\left\{\mu(s)\left(\frac{\partial^\alpha u(x,s)}{\partial s^\alpha}-R\left[x\right]\overline{u}(x,s)-q(x,s)\right)\right\}(ds^\alpha).$$

where μ is the general Lagrange multiplier that can be defined optimally via variational theory (Inokuti et al, 1978) and $\overline{u}(x,t)$ is the restricted variation, that is, $\delta\overline{u}(x,t)=0$.

So, the solution for equation (9) using this method is obtained the iteration formula as follows:

$$u_{n+1}(x,t)=u_n(x,t)-\frac{1}{\Gamma(\alpha+1)}\times\int_0^t\left\{\frac{\partial^\alpha u_n}{\partial s^\alpha}-\frac{\partial^2 u_n}{\partial x^2}+(k-1)\frac{\partial u_n}{\partial x}-ku_n\right\}(ds^\alpha).$$

Then, by using that iteration formula , it will be obtained the approximation $u_1(x,t),u_2(x,t),\ldots,u_n(x,t)$.

So, the solution for equation (9) is given by

$$\max\left(e^x-1,0\right)E_\alpha\left(-kt^\alpha\right)+\max\left(e^x,0\right)\left(1-E_\alpha\left(-kt^\alpha\right)\right).$$

The research concluded that the fractional variational iteration method is powerful and efficient in finding analytical approximate solutions as well as numerical solutions. The result shows that the present method is in excellent agreement with exact solution. On the other example of the research, it got the exact solution in two iterations.

2.3 New Lagrange Multipliers

The new identification of the Lagrange multipliers by means of the Sumudu transform, is employed to obtain a quick and accurate solution to the fractional Black-Scholes equation with the initial condition for a European option pricing problem (Ghandehari and Ranjbar, 2014). The fractional derivatives is described in Caputo sense and the fractional integral using Rieman-Liouville definition. This method finds the analytical solution without any discretization or additive assumption. The analytical method has been applied in the form of convergent power series with easily computable components.

The variational iteration method is a modified general Lagranges multiplier method (Inokuti et al, 1978). This method is a modification of the general Lagrange multiplier method into an iteration method, which is called correction functional. The major problem of the variational iteration method is the correct determination of the Lagrange multiplier, when the method is applied to ordinary and fractional equations. It is difficult for one to use the integration by parts to derive the Lagrange multipliers explicitly. The research consider a new modification of variational iteration method which is based on the Sumudu Transform.

The equation that used in this research is equation (9). By Applying the Sumudu transform on both sides of (9), it will be obtained the following iteration formula:

$$V_{n+1}(x,u) = V_n(x,u) + \lambda(u)\left[u^{-\alpha}V_n(x,u) - u^{-\alpha}(x,0) + S\left(-\frac{\partial^2 v_n}{\partial x^2} - (k-1)\frac{\partial v}{\partial x} + kv_n\right)\right].$$

After the identification of a Lagrange multiplier $\lambda(u) = \dfrac{-1}{u^{-\alpha}}$, one can derive

$$V_{n+1}(x,t) = S^{-1}\left[v(x,0) - \left[\frac{1}{u^{-\alpha}}S\left(-\frac{\partial^2 v_n}{\partial x^2} - (k-1)\frac{\partial v}{\partial x} + kv_n\right)\right]\right].$$

the exact solution can be given in a compact form

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = \max\left(e^x - 1, 0\right)E_\alpha\left(-kt^\alpha\right) + \max\left(e^x, 0\right)\left(1 - E_\alpha\left(-kt^\alpha\right)\right).$$

2.4 Homo-Separation of Variables

Ghandehari and Ranjbar (2016) used modified homotopy perturbation method (MHPM) to find the exact solution of the the option pricing problems based on the fractional Black-Scholes equation. The new method is a combination of two well-established mathematical methods, namely, the homotopy perturbation method (HPM) and the separation of variables method. The homotopy perturbation method (HPM) is a series expansion method used in the solution of nonlinear partial differential equations. In general is proved the homotopy perturbation method (HPM) is a special case of the homotopy analysis method (HAM) by Sajid et al (2007). The HPM is a universal approach which can be used to solve both fractional ordinary differential equations as well as fractional partial differential equations.

By using homo-separation of variables, fractional partial differential equation to be solved is changed into fractional ordinary differential equation. The method using Mittag-Leffler, Riemann-Liouville fractional integral and derivative as a tools for calculate process. It is said that the method is intuitive and very easy to understand. New approach converts the fractional Black-Scholes equation into a system of ordinary differential equations (ODEs) and after that proceeds to solve the resulting ODE. Finally, the resulting homo-separation of variables method, which is analytical, can be used to solve equations with fractional and integer order with respect to time.

The general form of the equation after applying the HPM still obtained partial differential equation that is

$$D_t^\alpha\left(v_1(x,t)\right) = D_t^\alpha\left(u_0(x,t)\right) - L\left(v_0(x,t)\right) - N\left(v_0(x,t)\right) - f(x,t) = 0 \ , \qquad (11)$$

where L is linear operator and N is nonlinear operator

After some process equation (11) become

$$u(x,0)D_t^\alpha\left(c_1(t)\right) + u(x)D_t^\alpha\left(c_2(t)\right) = L\left(u(x,0)c_1(t) + u(x)c_2(t)\right) + N\left(u(x,0)c_1(t) + u(x)c_2(t)\right) + f(x,t) = 0 .$$

In this case, the partial differential equation is changed into an ODE, which simplifies the problem at hand. The exact solution of the ODE is found when the target unknowns $c_1(t)$ and $c_2(t)$ are compute. By using this method, the solution for equation (9) is

$$\max\left(e^x-1,0\right)E_\alpha\left(-kt^\alpha\right)+\max\left(e^x,0\right)\left(1-E_\alpha\left(-kt^\alpha\right)\right).$$

2.5 Block-pulse operational matrix method

Mehrdoust et al (2016) used block-pulse operational matrix algorithm to approximate the solution of the time-fractional Black-Scholes equation with the initial condition for a European option pricing problem.. The fractional derivative described by Caputo sense. Beside that, the research is using fractional derivative of a vector too. The block-pulse wavelet has been successfully employed to obtain the numerical solutions of the fractional Black-Scholes equation with boundary condition for a European call option pricing problem. They presented the operational matrix of fractional order integration method to solve fractional Black-Scholes equation. This method transforms fractional differential equations into algebraic equations. The method could also be useful for solving other PDEs of mathematical finance.

2.6 Generalized differential transform method

In 2016, Yavuz and Ozdemir had used a new application of generalized two-dimensional differential transform method (GDTM) for solving time-fractional Black-Scholes option pricing equation (FBSE) with the initial condition for a European option pricing problem. The fractional derivative is described in the Caputo sense. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of ordinary or partial differential equations (Yavuz and Ozdemir, 2016). The generalized two-dimensional differential transform of the function $u(x,y)$ is as follows:

$$U_{\beta,\alpha}(p,h)=\frac{1}{\Gamma(\beta p+1)\Gamma(\alpha h+1)}\left[(D_{x_0}^{\beta})^p(D_{y_0}^{\alpha})^h u(x,y)\right]_{(x_0,y_0)},$$

where $(D_{x_0}^{\beta})^p=D_{x_0}^{\beta}.D_{x_0}^{\beta}.\cdots.D_{x_0}^{\beta}$, *p*-times, , is the so-called sequential fractional derivative (Podlubny, 1999).

By applying the generalized two-dimensional differential transform to both sides of equation (9), the libear time-fractional equation (9) transforms to

$$U_{1,\alpha}(p,h+1)=\frac{\Gamma(\alpha h+1)}{\Gamma(\alpha(h+1)+1)}\left[(p+1)(p+2)U_{1,\alpha}(p+2,h)+(k-1)(p+1)U_{1,\alpha}(p+1,h)-kU_{1,\alpha}(p,h)\right],$$

after some manipulations then obtained the series solution of equation (9) $v(x,t)$ for $x\geq 0$ in the form:

$$v(x,t)=\left(x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots\right)+(k)\frac{t^\alpha}{\Gamma(\alpha+1)}+(-k^2)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+(k^3)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots.$$

In a closed form, obtained

$$v(x,t)=\left(e^x-1\right)-\left[-1+\sum_{n=0}^{\infty}\frac{\left(-kt^\alpha\right)^n}{\Gamma\left(\alpha n+1\right)}\right]=e^x-E_\alpha(-kt^\alpha).$$

2.7 Laplace Homotopy Analysis Method (LHAM)

Yavuz and Ozdemir (2018) demonstrated a novel approximate-analytical solution method, which is called the Laplace homotopy analysis method (LHAM) using the Caputo–Fabrizio (CF) fractional derivative operator. The method is obtained by combining Laplace transform (LT) and the homotopy analysis method (HAM). They used the fractional operator suggested by Caputo and Fabrizio in 2015 based on the exponential kernel. They considered the LHAM with this derivative in order to obtain the solutions of the fractional Black–Scholes equations (FBSEs) with the initial conditions. According to the results of this study, it can be concluded that the LHAM in the sense of the CF fractional derivative is an effective and accurate method, which is computable in the series easily in a short time.

This new fractional operator has a smooth kernel that takes on two different impressions for the spatial and temporal variable. They had demonstrated the efficiencies and accuracies of the suggested method by applying it to the FBS option pricing models with their initial conditions satisfied by the classical European vanilla option. By using the real market values from the finance literature, it can obtaiedn how the option is priced for fractional cases of European call option pricing models. The successful applications of the proposed model prove that this model is in complete agreement with the corresponding exact solutions. Furthermore, the method is much easier than other homotopy methods, so the Laplace Transform allows one in many positions to eliminate the inadequacy essentially caused by insufficient conditions, which take part in other approximate-analytical methods like homotopy perturbation method (Madani et al, 2011).

The solution for equation (9) is given by

$$v(x,t) \approx v_n(x,t) = L^{-1}\left\{T_n(x,s)\right\} = \max\left(e^x - 1, 0\right) + \left(e^x - \max\left(e^x - 1, 0\right)\right)\left[\frac{e^{\frac{k\alpha t}{k\alpha - k - 1}} + k\alpha - k - 1}{k\alpha - k - 1}\right].$$

3. Conclusion

This paper reviews the recent advances made in the Black-Scholes model and solution methods. Many researchers have attempted to obtain the solution of the Black-Scholes model analytically or numerically, thereby adopting and using various direct and iterative methods, respectively. There are a few methods that has the same solution in the end. For LHPM, the main advantage of this method is to overcome the deficiency that is mainly caused by unsatisfied conditions. Thus, it is said that the LHPM methodology is very powerful and efficient in finding approximate solutions as well as numerical solutions. The fractional variational iteration method with Riemann-Liouville derivative shows that the present method is in excellent agreement with exact solution. It is proved with getting the exact solution in two iterations. A new lagrange multiplier whicih the modification of variational iteration method that is based on Sumudu Transform finds the analytical solution without any discretization or additive assumption. The analytical method has been applied in the form of convergent power series with easily computable components. Fractional partial differential equation to be solved is changed into fractional ordinary differential equation by using Homo-Separation of variables. It will be more easily if the equation is in the form of ordinary differential equation. So also with other methods that has their own efficiency and simplicity.

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References

Al Saedi, Y.M.,Tularam. G.A. A Review of the Recent Advances Made in the Black-Scholes Models and Respective Solutions Methods . *Journal of Mathematics and Statistics*. 14(1):29-39. 2018.

Black, F. and Scholes, M., The pricing of options and corporate liabilities, *The Journal of Political Economy*, 81: 637-659, DOI: 10.1086/260062, 1973

Chen, W, *Numerical Methods for Fractional Black-Scholes Equations and Variational Inequalities Governing Option Pricing*, School of Mathematics and Statistics, 2014.

D. D. Ganji and A. Sadighi, Application of homotopyperturbation and variational iteration methods to nonlinear heat transfer and porous media equations, *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 24–34, 2007.

Elbeleze, A.A., Kilicman, A., Taib, B.M., Fractional Variational Iteration Method and Its Application to Fractional Partial Differential Equation, *Mathematical Problems in Engineering*. http://dx.doi.org/10.1155/2013/543848, 2013.

G. C. Wu and E. W. M. Lee, Fractional variational iteration method and its application, *Physics Letters*, vol. 374, no. 25, pp. 2506–2509, 2010.

Ghandehari, M.A.M., Ranjbar, M, European option pricing of fractional Black-Scholes model with new Lagrange multipliers, *Computational Methods for Differential Equations*. vol. 2, no. 1, pp. 1-10, 2014.

Ghandehari, M.A.M., Ranjbar, M, Using Homo-Separation of Variables for Pricing European Option of the Fractional Black-Scholes Model in Financial Markets, *Mathematical Sciences Letter*, vol 5, no. 2, 181-187, 2016.

H. Ozer, Application of the variational iteration method to the boundary value problems with jump discontinuities arising in solid mechanics, *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 4, pp. 513–518, 2007.

Hackmann, D., Solving the Black Scholes Equation using a Finite Difference Method, Available online : math.yorku.ca/~dhackman/BlackScholes7.pdf , 2009.

Harun, H.F. and Hafizah, M., Implied adjusted volatility functions: Empirical evidence from Australian index option market, *AIP Conf. Proc*., 1643: 622-627, DOI: 10.1063/1.4907503, 2015.

He, J. H., Application of homotopy perturbation method to nonlinear wave equations, *Chaos Solitons & Fractals*, 26, pp. 695–700, 2005

He, J. H., Some asymptotic methods for strongly nonlinear equations, *International Journal Modern Physics*. B20, pp. 1141–1199, 2006.

Hirsa. A., Madan, D.B., Pricing American options under variance gamma, *Journal of Computational Finance*, 7(2):63-80, 2004.

Hull, J., S. Treepongkaruna, D. Colwell, R. Heaney and D. Pitt, 2013. *Fundamentals of futures and options markets*. Pearson Higher Education AU.

J. Hull and A. White., The pricing of options on assets with stochastic volatilities, *The journal of finance*, 42(2):281-300,1987.

Kou, S.G., A jump diffusion model for option pricing, *Management science*, 48(8):1086-1101, 2002.

Kumar, S., Yildrim, A., Khan, Y., Jafari, H., Sayevand, K., Wei, L., Analytical Solution of Fractional Black-Scholes European Option Pricing Equation by Using Laplace Transform, *Journal of Frational Calculus and Applications*, Vol 2, No. 8, pp 1-9, 2012.

M. Inokuti, H. Sekine, and T. Mura, General use of the Lagrange multiplier in non-linear mathematical physics, in *Variational Method in the Mechanics of Solids*, S. Nemat-Nasser, Ed., pp. 156–162, Pergamon Press, Oxford, UK, 1978.

M. Sajid, T. Hayat and S. Asghar, Comparison between the HAM and HPM solutions of thin film flows of non Newtonian fluids on a moving belt, *Nonlinear Dynamic*, vol 50, 27-35, 2007.

Madani, M., Fathizadeh, M., Khan, Y., Yildirim, A., On the coupling of the homotopy perturbation method and Laplace transformation, *Mathematical Computation Model*, 53, 1937–1945, doi:10.1016/j.mcm.2011.01.023, 2011.

Mehrdoust, F., Refahi, A.H., Mashoof, M., Hassanzadeh, S, Block-pulse operational matrix method for solving fractional Black-Scholes equation, *Journal of Economic Studies*, https://doi.org/10.1108/JES-05-2016-0107, 2016.

Podlubny, I., *Fractional differential equations*, Academic Press, New York, 1999.

Wilmott, P., Howison. S, and Dewynne, J., *The mathematics of financial derivatives: A student introduction*, Cambridge University Press, 1995.

Yavuz, M., Ozdemir, N., Okur, Y.Y, European Vanilla Option Pricing Model of Fractional Order without Singular Kernel, *Fractal and fractional*, vol 2, no 3, 2018.

Yavuz, M., Ozdemir, N., Okur, Y.Y., Generalized differential transform method for fractional partial differential equation from finance, *International Conference on Fractional Differentiation and its Applications*, Novi Sad, Serbia, July 18 - 20, 2016.

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