Maximal Queen Placements for the Mod 2 *n*-Queens Game

Cindy Qiao
A.Y. Jackson Secondary School
Ontario, Canada

Abstract

The century old problem of configuring n queens on a chessboard so that none of them attack one another, known as the "*n*-queens problem", has been studied intensively by researchers, along with many variants. To this day, the problem stands as a prominent example for backtracking search methods. Its demonstration of constraint satisfaction, as well as systematic and heuristic search methods, highlights its utility in fields such as artificial intelligence (AI) and program development. This paper focuses on a contemporary variant, the "mod 2 *n*-queens problem", recently proposed by Brown and Ladha. This paper uses graph theory to solve some of the open problems Brown and Ladha posed.

Keywords
Chess, modulo n-queens, domination, independence, queen graph.

1. Introduction
The Mod 2 *n*-Queens Game
The *n*-queens problem is a challenge to place n queens on a $n \times n$ chessboard so that none of these queens attack one another. Noon describes this problem as a two-player game, where players successively place queens on a $n \times n$ board so that none attack each other (Noon 2002). The "mod 2 *n*-queens game", rooted in the ideas of the *n*-queens game, was introduced by Brown and Ladha (Brown and Ladha 2019): on a $n \times n$ board, a queen can only be placed on a square if there are an even number of queens attacking that square, and a queen cannot be placed on a square if there are an odd number of queens attacking the square. These squares are respectively defined as open and closed squares. With a square taking on a value of the number of queens attacking it, these rules translate mathematically in terms of modulo 2: any square with an even integer value would be congruent to 0 mod 2, and any with an odd integer value would be congruent to 1 mod 2. In a mod 2 *n*-queens game, two players take turns placing queens, either cooperatively to fill the board, or competitively until queens can no longer be placed.

To briefly compare the *n*-queens game to the mod 2 *n*-queens game, we see that a maximum of n queens can be placed in the n-queens game on a $n \times n$ board when $n = 1$ or $n \geq 4$, whereas a total of n^2 queens can be placed in the mod 2 *n*-queens game on a $n \times n$ board for all positive odd integers n. Figure 1 compares the squares that are left open after a placement of five queens on a 5×5 board in the *n*-queens game and the mod 2 *n*-queens game. To generalize, more squares are opened for the placement of queens in the mod 2 *n*-queens game, and the bound on the maximum number of queens that can be placed is thus increased.

In addition, the order of the placement of queens is relevant to the mod 2 *n*-queens game, but not to the *n*-queens game. This logic stands as the parity of the board changes with every queen placed. These slight, but notable differences between the two games poses many more questions and ideas for exploration in the mod 2 *n*-queens game.

There are several of which Brown and Ladha call game states encompassed within the mod 2 *n*-queens game for an arrangement of queens on a $n \times n$ board. Rather than developing strategies to win the mod 2 *n*-queens game, this paper focuses on these game states which we treat as problems within the mod 2 *n*-queens game. Specifically, we explore complete and locked boards for a $n \times n$ board. Configurations found in these game states can provide a basis for future studies that hope to define a winning strategy in the mod 2 *n*-queens game.

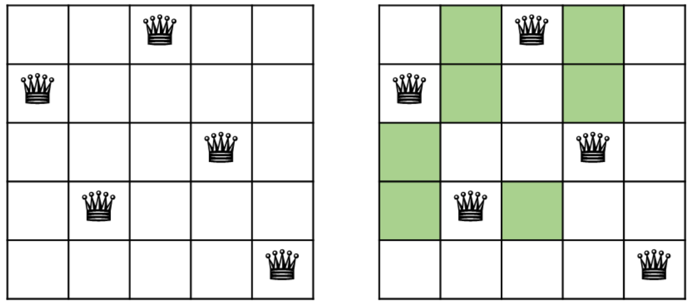

Figure 1: Left: A maximum of 5 queens placed on a 5 × 5 board in the *n*-queens game. Right: Squares highlighted in green indicate open squares after the same placement of 5 queens on a 5 × 5 board in the mod 2 *n*-queens game.

Definitions & Terminology

Most of the terminology used to describe the mod 2 *n*-queens game in this paper is shared with those previously defined by Brown and Ladha.

A board is said to be **complete** if all n^2 squares are filled with queens. A **locked** board is one that has fewer than n^2 queens, but no more legal moves remain.

In order to describe placements, squares on the $n \times n$ board will be indexed by ordered pairs (i,j), where $1 \leq i \leq n$ indicates rows numbered from top to bottom, and $1 \leq j \leq n$ indicates columns numbered from left to right. Diagonals running in the left to right and bottom to top direction are described when the sum of the indices of each square is k for some integer $1 \leq k \leq 2n$, whereas diagonals running in the right to left and top to bottom direction are described when the difference of the indices of each square is k for some integer $-(n - 1) \leq k \leq n - 1$. We call these the k-sum diagonal and the k-difference diagonal, respectively. We also call the $(n + 1)$-sum diagonal the main sum diagonal, and the 0-difference diagonal the main difference diagonal.

Motivation for Mod 2 n-Queens Research

Currently, the n-queens problem is one that can be solved using a constraint programming approach that utilizes propagation and backtracking (The N-queens Problem 2020). As the mod 2 *n*-queens game is an extension of the *n*-queens game, researching the mod 2 *n*-queens game gives us more insight into solving constraint satisfaction problems using systematic and heuristic search techniques, which is ultimately useful in the fields of artificial intelligence (AI) and program development.

Our Results

This paper proves two open problems posed in Brown and Ladha's exploration of the mod 2 *n*-queens game.

In the complete game state, Brown and Ladha (Brown and Ladha 2019) proved that a total of n^2 queens cannot be achieved on $n \times n$ boards in which n is an even positive integer. We present our own distinctive proof on the same proposition using a graphical representation of the mod 2 *n*-queens game.

Brown and Ladha (Brown and Ladha 2019) conjecture that on such a board, there exists a legal placement of $n^2 - 2$ queens. We build upon the construction steps Brown and Ladha had outlined for the placement of n^2 queens on a $n \times n$ board in the odd case and present a proof for the placement of $n^2 - 2$ queens on any even-sized $n \times n$ board.

We also present a proof for the placement of $n^2 - 2$ queens being the maximally locked position for any even-sized $n \times n$ board, thus answering Brown and Ladha's question of whether a total of $n^2 - 2$ queens is the maximum number of queens which can be placed on an even-sized $n \times n$ board.

2. Literature Review

Brown and Ladha's Results on The Mod 2 *n*-Queens Game

Brown and Ladha's study of the mod 2 *n*-queens game describes some basic and important results. A number of these results are relevant to this paper as the same concepts and ideas previously uncovered can be applied to the open problems we are concerned with.

In Brown and Ladha's (Brown and Ladha 2019) exploration of complete boards, they proved that the maximum number of n^2 queens is achievable through legal play, but only for boards of odd sizes. For even-sized boards, Brown and Ladha (Brown and Ladha 2019) inductively proved that a complete solution of n^2 queens cannot be achieved through legal gameplay.

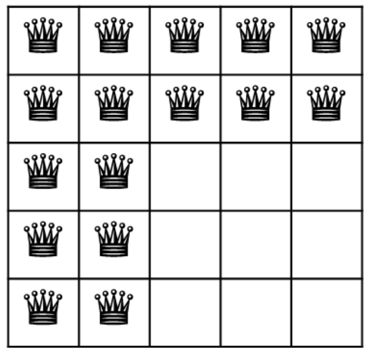

Figure 2: A 5 × 5 board with queens placed in the top two rows and leftmost two columns.

Brown and Ladha (Brown and Ladha 2019) describes the idea of first filling the top two rows and leftmost two columns to achieve an unfilled $(n-2) \times (n-2)$ board in the lower right corner. Critically, each unfilled square is attacked by an even number of queens. Each square is attacked vertically, horizontally, and along the difference diagonal by exactly two queens on each line. Squares on the main sum diagonal or to the left of it are attacked by four more queens on the sum diagonal, and squares on the $n + 2$ sum diagonal are attacked by exactly two more queens. All other squares are attacked by zero more queens. Overall, all the squares on this $(n-2) \times (n-2)$ board can be observed to be open squares. From this observation, Brown and Ladha state that gameplay on the board shown in Figure 2 is correspondent to gameplay on the $(n-2) \times (n-2)$ board with no queens and that they can inductively proceed to a 1 × 1 board in the lower right corner. The 1 × 1 board can then be filled with a queen to achieve a n^2 queens on a $n \times n$ board where n is an odd positive integer.

A sequence of legal moves is presented by Brown and Ladha (Brown and Ladha 2019) to show that it is indeed possible to achieve the configuration of queens as shown in Figure 3.

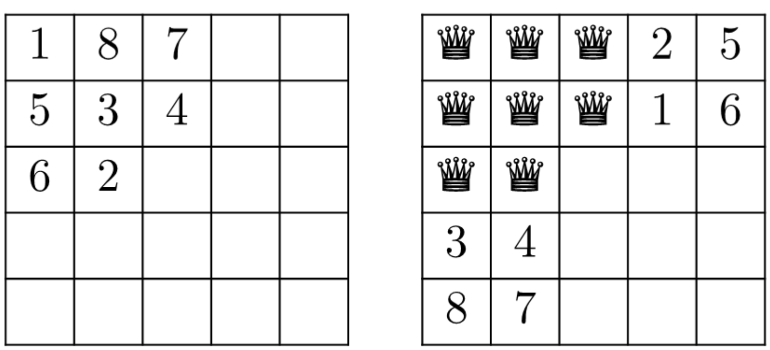

Figure 3: Left: First eight queens are placed. Right: Next eight queens are placed to fill the top two rows and leftmost two columns for a 5 × 5 board.

The process outlined in Figure 3 is repeated on the $(n-2) \times (n-2)$ board until the board is reduced to the last unfilled square in the lower right corner which can then be filled with a queen. Brown and Ladha's process for filling in the top two rows and leftmost two columns become an idea that this paper reuses in proving that an even-sized $n \times n$ board can always be filled with $n^2 - 2$ queens.

We now move on to briefly describe Brown and Ladha's exploration of the locked game state, and its relevance to this paper. Brown and Ladha's results on configurations in the locked game state provides a deeper understanding and insight into unanswered questions regarding locked states. In the mod 2 n-queens game, locked states are explored for strategic purposes; players want to avoid these positions themselves while leading their opponent into them.

Immediately, we note that $n \times n$ boards with $n = 1$, 2, or 3 can be simply locked with the placement of one queen; however, when $n > 3$, this rule cannot be enforced any further. Similar to their explorations with the complete game state, Brown and Ladha separate their exploration of locked boards into even and odd states and provide a consistent method that can be used to lock each.

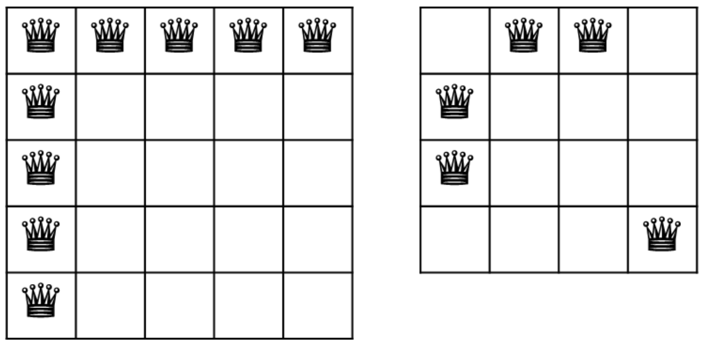

Figure 4: Left: A locked 5 × 5 board. Right: A locked 4 × 4 board.

Indeed, the placement of queens for an odd board as shown in Figure 4, described as the set $\{(1, i),(i, 1)|1 \leq i \leq n\}$, locks the board. Brown and Ladha's (Brown and Ladha 2019) proof for this follows similarly to the previously discussed case in the complete state, with each unfilled square being attacked by an odd number of queens, hence, closed. The same reasoning is applied for the placement of queens for an even-sized $n \times n$ board, described as the set $\{(1, i),(i, 1)|2 \leq i \leq n - 1\} \cup \{(n, n)\}$ when $n > 2$. Once these configurations satisfy the condition of locking the odd and even boards, it is critical that it is possible to reach these configurations through a set of legal moves.

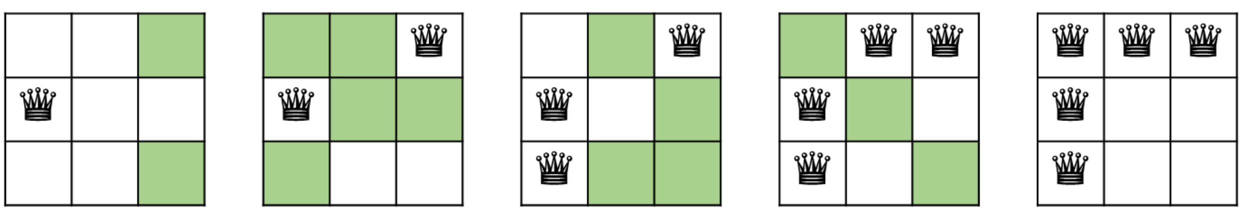

Figure 5: A sequence for the placement of queens to reach a locked state for the 3 × 3 board.

To extend the gameplay as shown in Figure 5 to odd boards of larger dimensions, the strategy is to repeat the first four moves: first in the two adjacent unfilled squares in the top row, then in the two adjacent unfilled squares in the leftmost column until the locked state is reached. For even boards, the same first four steps and strategy as in the odd case is followed until the last step, where the last queen will be placed in the lower right corner instead.
With Brown and Ladha's technique for locking boards with even and odd parity, there is now an upper bound on number of queens required to lock boards. However, their paper goes on to ask if fewer queens can lock boards of larger sizes, and the minimum number of queens needed to lock a $n \times n$ board.

Queen Graphs
The placement of queens on a board can be represented through means of a graph, and we call such a graph a "Queen Graph" (Weisstein 2021b). Figure 2 presents some simple queen graphs for $2 \times n$ boards of various sizes. For a $m \times n$ board, we can have the queen graph $Q_{m,n}$ with mn vertices, where each vertex represents a square on the $m \times n$ board.

Edges are drawn between vertices that attack each other through a queen's move, and vertices joined by an edge are said to be adjacent. The degree of a vertex is the number of edges going into that vertex.

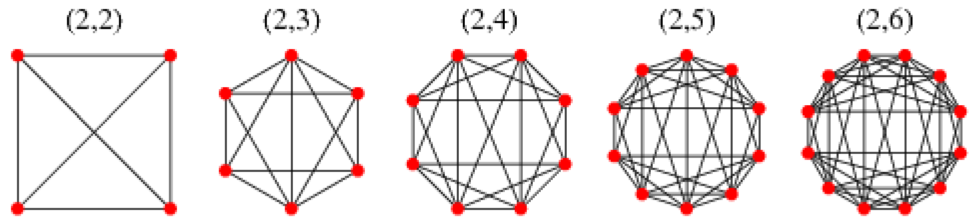

Figure 6: Queen graphs for $2 \times n$ boards of various sizes. Source: "Queen Graph" from Wolfram MathWorld. [8]

In this paper, we also look at induced subgraphs within a general queen graph. For such induced subgraphs, we only use vertices to represent placed queens on a board, and we draw edges between vertices when two queens attack each other.

Domination in Queen Graphs

Domination numbers and sets have been previously studied in various queen placement problems. The *queen's domination problem* is one that asks for the minimum number of queens that can be placed on a board so that all the squares on a board are covered. Proposed in 1862 by de Jaenisch (de Jaenisch 1862), the queen's domination problem was one of the first known problems to consider domination. In the queen's domination problem, configurations are be said to be *non-attacking* if placed queens do not attack one another.

In the queen's domination problem, we say that a vertex ***dominates*** itself and any adjacent vertices (Burchett 2005). A queen graph, Q, is said to be dominated by a subset of vertices, S, if any vertex in Q is dominated by a vertex in S. The minimum number of queens needed to dominate a given $n \times n$ board, which we denote as $\gamma(Q_n)$, is known as the ***domination number*** of the queen graph (Burchett 2005). A ***dominating set*** is the subset of vertices, S, within the larger set of vertices in a graph with every vertex not in S adjacent to at least one of the vertices within S (Bray and Weisstein 2021).

In a standard, 8×8 chessboard, it has been proven that $\gamma(Q_8) = 5$. Much progress has been made on this problem, with Rouse Ball (Burger et al. 1994) presenting minimum dominating sets for $n \leq 8$ in Q_n in 1892, followed by Ahrens (Ahrens 1910), who in 1910 provided minimum dominating sets of Q_n for $9 \leq n \leq 13$ and $n = 17$.

As the idea of domination in queens placement problems have been previously proposed and studied, it makes sense to apply the same basic ideas of domination numbers and sets in graph theory to the defined queen graph for the mod 2 n-queens game. In this way, finding the domination number is equivalent to answering a question about locking the board with the smallest number of queens. The minimum domination set can provide an idea of how to place such queens. Algorithms have been developed to attempt at finding such domination numbers and sets in graphs, but with the queen's domination problem classified as NP-complete, it becomes impractical to rely on computer searches and algorithms.

Independence in Queen Graphs

In the n-queens problem, the idea of independence applies when trying to find the maximum number of queens that can be placed on a $n \times n$ board such that no two queens attack each other. In 1979, this problem has already been answered by Madachy (Madachy 1979): $n - 1$ queens can be placed when $n = 2, 3$, and n queens can be placed for all other values of n. In 1874, Pauls (Pauls 1874) appears to have provided the earliest proof that a placement of n queens can always be achieved on a $n \times n$ board in the n-queens problem. Following Pauls, various other authors have published their own distinct proof to the same problem.

An *independent vertex set* of a queen graph, Q, is a subset of the vertices in which no two vertices in the subset is connected by an edge. We call the cardinality of the largest independent vertex set the ***independence number*** (Weisstein 2021a). In the mod 2 n-queens game, we can look for the largest independent vertex set as well as the independence number to describe the maximum number of queens that can be placed on a $n \times n$ board. For a $n \times n$

board where n is an odd positive integer, Brown and Ladha have already found the independence number to be n^2; however, the question of the independence number remained for $n \times n$ boards of even sizes. In this paper, we find that this independence number turns out to be $n^2 – 2$ and describe how to achieve the largest independent set.

3. Methods

This paper uses the concepts of 'Queen Graphs', as well as domination and independence in queen graphs to prove several open questions proposed by Brown and Ladha.

4. Results and Discussion

Placement of $n^2 - 2$ Queens on Even-sized Boards

We have seen Brown and Ladha's (Brown and Ladha 2019) proof of being able to place n^2 queens on a $n \times n$ board where n is an odd positive integer and how it is impossible to achieve the same results when n is an even positive integer. Brown and Ladha (Brown and Ladha 2019) then proposed that $n^2 - 2$ queens can be legally placed on a $n \times n$ board when n is an even positive integer. We note that this can only be true when n is an even positive integer greater than or equal to 4, as when $n = 2$, a maximum of one queen can be placed to achieve a total of $n^2 - 3$ queens.

A brief explanation is provided on how to construct this placement of $n^2 - 2$ queens, claiming that "One such construction could follow similarly to the odd case by applying induction and filling the topmost two rows and leftmost two columns,". Referring to figure 7, if we look at filling the topmost two rows and leftmost two columns in the 4×4 board, it becomes clear that the last four squares in the lower right corner are all open, and the placement of one more queen on any of these squares would lock the last three, leaving us with $n^2 - 3$ queens placed in total.

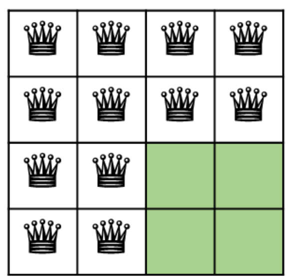

Figure 7: Topmost two rows and leftmost two columns filled with queens on a 4 × 4 board.

Although Brown and Ladha's construction idea fails immediately for the 4 × 4 board, we further explore their idea for boards of larger sizes.

Proposition 1. If n is an even positive integer with $n \geq 4$, $n^2 - 2$ queens can be placed legally on a $n \times n$ board.

Proof. We apply the idea of filling the topmost two rows and leftmost two columns to boards of larger sizes, until an empty 4 × 4 board is reached (refer to figure 8). We know the empty 4 × 4 board in the lower right corner of the board can always be achieved as filling the topmost two rows and leftmost two columns do not change the parity of the unfilled squares. Brown and Ladha have already provided the process for filling in the topmost two rows and leftmost two columns in their proof for achieving n^2 queens on an odd-sized board, so we simply follow their outlined process.

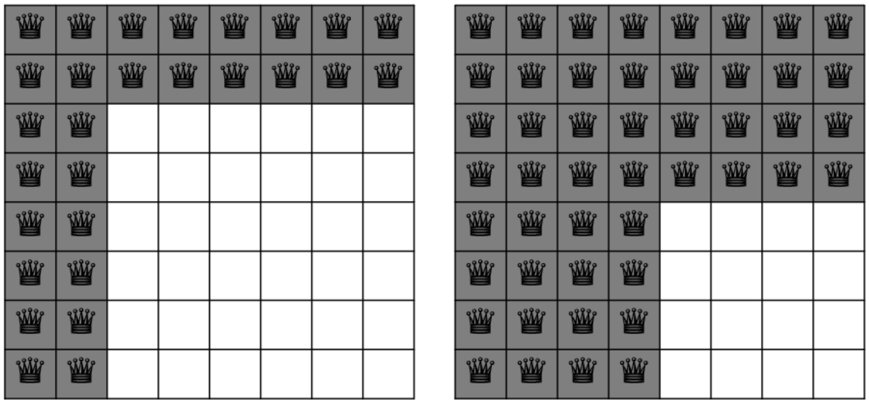

Figure 8: Left: Topmost two rows and leftmost two rows are filled for an 8 × 8 board. Right: Next two top rows and leftmost columns are filled to leave an empty 4 × 4 board.

For the 4 × 4 board, we then suggest a sequence of legal moves to lock the board with two unfilled squares on $(n - 3, n)$ and $(n - 1, n - 1)$ as shown in Figure 9. In turn, this locks the even-sized $n \times n$ board with $n^2 - 2$ queens. We know this sequence of moves can always be applied because the parity of the squares in the 4 × 4 board will stay the same after the previous placement of queens. Hence, we have achieved a legal placement of $n^2 - 2$ queens on a $n \times n$ board where n is an even positive integer.

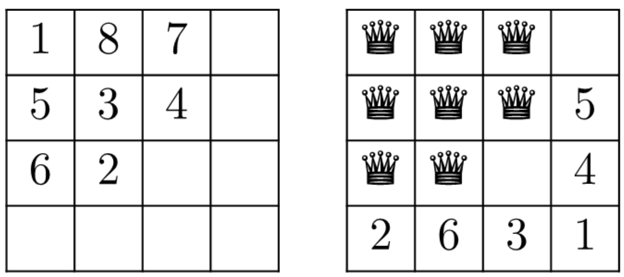

Figure 9: Left: First eight queens are placed in 4 × 4 board. Right: Next six queens are placed in 4 × 4 board to leave two empty, locked squares.

Knowing that $n^2 - 2$ queens can always be legally placed on a $n \times n$ board where n is an even positive integer and $n \geq 4$, the question arises of whether this is the maximum number of queens which can be placed on an even-sized board. This is also a question of finding the maximally locked solution for even-sized boards which this paper explores and answers next.

Maximally Locked Solution for Even-sized Boards

Brown and Ladha (Brown and Ladha 2019) utilized empirical data as well as computer simulation to verify that for a $n \times n$ board, when $n = 4$, $n^2 - 2$ queens is the maximum number of queens which can be legally placed. However, computational power is modest and fails to compute for larger values of n in a reasonable time span. For $n = 6$, it would take approximately 22 years to verify whether a total of $n^2 - 2$ queens is the maximum number of queens which can be legally placed, so computational power and algorithms are not effective in this case.

We now define ***"even-neighbour labelling"*** as being able to number a vertex in a queen graph only when it is adjacent to an even number of vertices that have been previously numbered. This is likewise to how we can only place a queen on a square attacked by an even number of queens previously placed. When we assign numbers 1 to n to all the vertices in a queen graph, we call this a ***complete labelling***. If all the vertices are numbered with even-neighbour labelling in consideration, we can call this a ***complete even-neighbour labelling***. Therefore, a queen graph with a complete even-neighbour labelling corresponds to being able to legally place a number of queens (equivalent to the number of vertices within the queen graph) on a board.

Going back to the problem at hand, asking the question of whether $n^2 - 2$ queens is the maximal solution on $n \times n$ boards where n is an even positive integer is essentially asking a question of whether $n^2 - 1$ queens can be achieved through legal gameplay on the defined $n \times n$ board. We already know a complete solution on even-sized boards is unachievable from Brown and Ladha's inductive proof, though we now suggest another proof through the representation of an even-sized board filled with n^2 queens in terms of a queen graph.

Lemma 1. If n is an even positive integer, the $n \times n$ board does not have a complete solution of n^2 queens.

Proof. To prove that n^2 queens cannot be placed, we will show that the queen graph for n^2 queens on an even-sized $n \times n$ board does not have a complete even-neighbour labelling. We consider a $n \times n$ board where n is an even positive integer. Along the vertical and horizontal lines, each square attacks a total of $2(n - 1)$ squares which is an even number of squares, and each square also attacks an odd number of squares along one of the diagonals. In total, we see that each square attacks an odd number of squares. If we use vertices on a queen graph to represent every square, each of these vertices should in turn have an odd degree. If every vertex has an odd degree, it becomes clear that a complete even-neighbour labelling cannot be achieved, and that n^2 queens cannot be placed on an even-sized $n \times n$ board. We thus complete our distinctive proof.

Even though we know that a complete solution cannot be achieved, to prove that $n^2 - 2$ queens is the maximally locked solution on an even-sized board, we need to also prove that a legal placement of $n^2 - 1$ queens is impossible. To prove this, we first attempt to count the total number of edges in a n^2 queen graph.

Lemma 2. The total number of edges in a n^2 queen graph, where n is an even positive integer, has an even parity.

Proof. Fundamentally, we want to sum up the degrees of each of the individual n^2 vertices and then divide the resulting number by two to account for counting each edge twice (we know each edge is counted twice due to each edge going into two vertices). To carry this out, we first look at any $n \times n$ board where n is an even positive integer. Although we are looking at a board, we think of every square on this board as a vertex in a queen graph.

We see that each square attacks a total of $2(n - 1)$ squares along the horizontal and vertical lines. Within a ring, as we move from the corner square along the edge, a square's attack along the greater diagonal decreases by one while its attack along the lesser diagonal increases by one. This keeps the sum of the diagonals attacked by any square within the same ring equal to $n - 1$. Therefore, we notice that every square part of the same "ring" on such boards have the same degree, while squares on different rings have a different degree. For example, in any $n \times n$ board, the innermost ring consisting of four squares all have a degree of $4n - 5$, whereas squares on the outermost ring each have a degree of $3n - 3$. We then make the critical observation that each ring can always be broken down into four pieces, as shown in Figure 10.

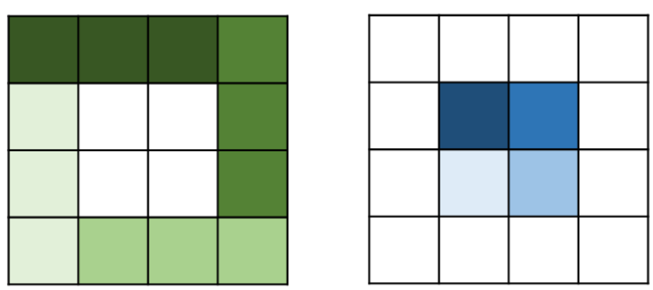

Figure 10: Left: Outermost ring of a 4 × 4 board broken into four pieces. Right: Innermost ring of a 4 × 4 board broken into four pieces.

Hence, the total number of squares in the outermost ring can be described as $4(n - 1)$, $4(n - 2)$ in the following ring, down to $4(n(n-1))$ in the innermost ring. To count the total number edges of the n^2 queen graph where n is an even positive integer, we sum up the degrees of each individual vertex in this graph and divide the total by two. The degree of each individual vertex proves to be insignificant. Vertices part of the same ring on a board carry the same degree, so we simply group such vertices together.

Keeping in mind that each ring can be broken into four pieces, we can derive that the sum of the degrees of each individual vertex in this n^2 queen graph is going to be a multiple of four. In order to ensure that the total number of edges in the n^2 queen graph is even, the sum of all the degrees of each individual vertex divided by two needs to be even. Even though the division of an even number by an even number does not always give us an even number, we know that the calculation $4 \div 2$ gives us two, an even number. Therefore, we can conclude that the total number of edges in a n^2 queen graph, where n is an even positive integer, has an even parity.

We now proceed to prove that one cannot achieve a placement of $n^2 - 1$ queens through legal gameplay on a $n \times n$ board where n is an even positive integer.

Proposition 2. If n is an even positive integer, $n^2 - 1$ queens cannot be placed legally on a $n \times n$ board.

Proof. We set up a proof by contradiction by assuming that there exists a legal placement of $n^2 - 1$ queens on a $n \times n$ board where n is an even positive integer and proceed to derive a contradiction. We consider this placement of $n^2 - 1$ queens as a list of vertices and append on the last position so that we have n^2 vertices. Hence, we now have a n^2 queen graph where n is an even positive integer.

In a complete labelling of this graph, we see that the number of edges in the graph can be totaled by counting the number of neighbours ahead of each vertex. When we assume that a placement of $n^2 - 1$ queens is possible, we are also claiming that it is possible to achieve even-neighbour labelling with $n^2 – 1$ vertices on the n^2 queen graph.

If we know it is impossible to achieve a complete even-neighbour labelling for the n^2 queen graph (refer to proof for Lemma 1), but assume it is possible to have even-neighbour labelling for $n^2 – 1$ vertices on the n^2 queen graph, we can conclude that one vertex in the n^2 queen graph will be adjacent to an odd number of vertices previous to it.
In total, we have $n^2 – 1$ vertices adjacent to an even number of vertices previous to it, and one vertex adjacent to an odd number of vertices previous to it. Recall that vertices adjacent to one another are joined by an edge. Mathematically, we can now sum up the total number of edges:

(n^2 – 1 vertices) × (even #) + (1 vertex) × (odd #) = even # + odd # = odd #

When assuming that there exists a legal placement of $n^2 – 1$ queens, our calculations tell us the total number of edges in the n^2 queens graph carries an odd parity. Here, we can draw a contradiction: in **Lemma 2**, we proved that the total number of edges in a n^2 queen graph, where n is an even positive integer, must be of an even parity; however a placement of $n^2 – 1$ queens only appears to be possible when the total number of edges in a n^2 queens graph has an odd parity. Therefore, we can conclude that it is impossible to achieve a placement of $n^2 – 1$ queens through legal gameplay on a $n \times n$ board where n is an even positive integer.

Exploring Lower Bounds on Locked Game States
With Brown and Ladha's (Brown and Ladha 2019) proof for a placement of queens that can lock boards of even or odd parity, it appears they have imposed an upper bound on the number of queens needed to lock an even or odd-sized board; for a $n \times n$ board, when n is odd, a maximum of $2n – 1$ queens is needed to lock the board, and when n is even, a maximum of $2n – 3$ queens is needed to lock the board. Brown and Ladha proceed to ask if the upper bound stated is a strict bound, and the question: "What is the minimum number of queens needed to lock a $n \times n$ chessboard?".

Their question of the minimum number of queens needed to lock a $n \times n$ chessboard goes back to a domination problem once we represent the problem in terms of a queen graph. We would want to find the domination number of a board represented as a queen graph and be able to do this with boards of larger sizes as well. For example, the domination number of a 3 × 3 board would be one; one queen is placed in the center of the board to dominate all squares (vertices).

Particularly for the mod 2 n-queens game, it becomes much harder to find domination numbers in graphs with numerous vertices due to the additional rules of the game. Nevertheless, we notice that the domination problem in the mod 2 n-queens game is similar to the queen's domination problem earlier discussed in this paper, as the key idea summarizes to finding domination numbers and sets within graphs. Keeping in mind the NP-completeness of the queen's domination problem, it is reasonable to assume that it will be notably difficult to find domination numbers and sets for the mod 2 n-queens problem, especially for boards of larger sizes. However difficult, we cannot claim this problem to be impossible to solve, so it remains an open question for future research.

Even though we cannot disprove Brown and Ladha's stated upper bounds for all boards, we can disprove the $2n - 1$ queens upper bound for the specific 5 × 5 board.

Proposition 3. For a 5 × 5 board, there exists a placement of $2n - 3$ queens which can be achieved through legal gameplay and locks the board.

Proof. Refer to figure 11.

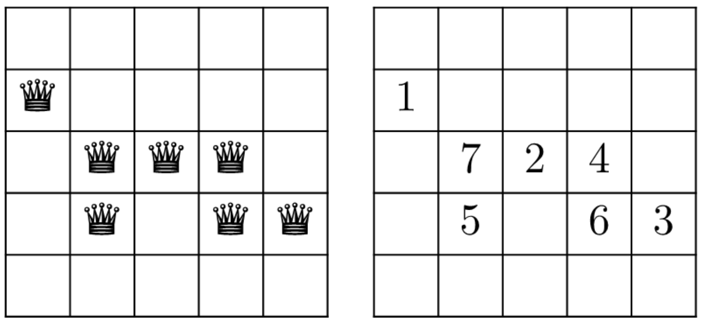

Figure 11: Left: Seven queens placed on a 5 × 5 board, locking the board. Right: Sequence of legal moves to place seven queens on a 5 × 5 board.

The configuration of queens in the 5 × 5 board does not have any symmetry or notable patterns, so applying the same configuration to larger boards is not practical.

5. Conclusion

In summary, distinct and detailed proofs were outlined in this paper to answer two of the open problems posed in Brown and Ladha's original exploration of the mod 2 n-queens game. Using the concepts of queen graphs, domination, and independence, this paper has taken a unique approach to analyzing the mod 2 n-queens problem. Such an approach may be applied in future studies of similar games or problems that are concerned with constraint satisfaction.

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Biography

Cindy Qiao is a student heading into her first year at the University of Toronto to study computer science.

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