Mathematical programming formulation for large-scale standing-wave thermo-acoustic refrigerator design optimization

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Abstract

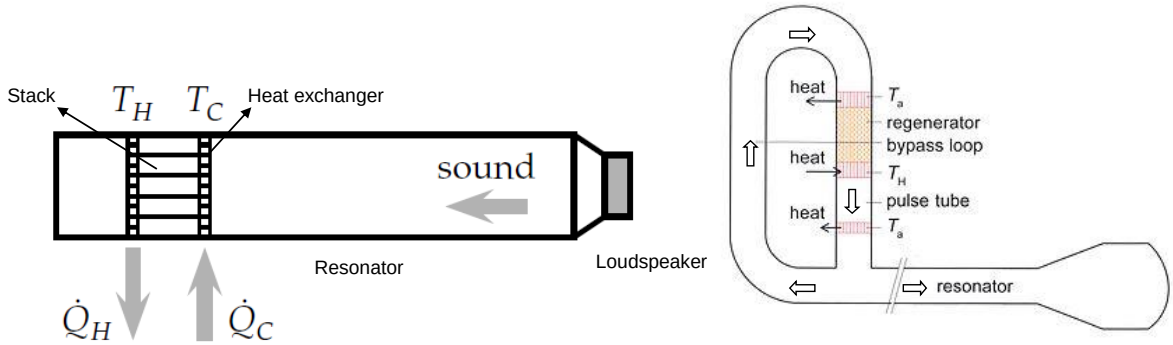
Thermo-acoustic refrigerator constitutes a viable alternative to the search for suitable replacement of conventional refrigeration systems. Although the device is simple to build and environmentally friendly, the issue related to its efficiency requires some investigation. Advanced optimization schemes are being developed in order to handle the designing of the device which appears to be challenging with regards to the interdependence of its design parameters. In this paper, a mathematical programming formulation is presented in order to model and optimize the geometry of a simple standing-wave thermo-acoustic refrigerator. The model developed has been implemented in the software GAMS. The geometry of the stack, described as the heart of the device, constitute the focus of this study. The stack length and the stack position are the main variables considered in this work. This work provides guidance on the selection of the most suitable geometry of the stack in order for the device to achieve a relatively higher performance. In addition, the details of the mathematical programming model have been disclosed for further analysis and improvement.

Keywords

Thermo-acoustic, refrigeration, optimization, GAMS;

1. Introduction

The field of refrigeration and air-conditioning is impacted greatly by the concerns arising from the issue related to the ozone depletion and global warming due to the chlorofluorocarbons (CFCs) contained in refrigerants. The proposal to adopt hydrochlorofluorocarbons (HCFCs) or hydrofluorocarbons (HFCs) as a replacement of CFCs may not solved this problem permanently. It appears that developing new technologies that incorporate ammonia, hydrocarbons, water, air, carbon dioxide and air, as alternatives to natural refrigerants, may provide a long-lasting solution (Ciconkov and Zahid, 2009). Thermo-acoustic refrigerators is potentially a clean technology that can be considered because they don't require the use of harmful refrigerants during operation. The devices normally consist of a sound wave that interact with a porous media, inducing cooling due the thermo-acoustic effect within the device. With suitable pressure, geometrical configuration of the device and proper operation setting, a significant cooling could be achieved. Based on the shape of the device, thermo-acoustic refrigerators can be classified in two different categories: Standing-wave and travelling wave devices. The former is generally smaller, simple to build and relatively cheaper but could only exhibit a relatively lower coefficient of performance (COP). Standing-wave thermo-acoustic refrigerator consists a long tube (or resonator) filled with inert gas containing a porous media refer to as stack. This porous media has a low thermal conductivity and high specific heat capacity. In order to generate the thermo-acoustic effect, a relatively higher temperature gradient is applied across the extremities of the stack through two heat exchangers. A loudspeaker is generally used to generate the sound as shown in Figure 1(a). On the other hand, travelling-wave device consists of a toroidal tube and a regenerator sandwiched between two heat exchangers. As a results of the topological feature of the toroidal tube, the phasing between the travelling acoustic wave and the pressure and velocity within the device is the same. Hence, the travelling-wave thermo-acoustic device denomination. Travelling-wave devices are generally more efficient in comparison to their standing-wave counterparts. A typical travelling-wave device is shown in Figure 1(b). This looped tube incorporates a regenerator having smaller porosity as compared to the stack.



(a) (b)

Figure 1. Simple standing and travelling-wave thermo-acoustic refrigerator

Despite the existence of several working devices, there is a lot of focus on improving the performance of the system in order for thermo-acoustic refrigeration to be accepted by the general public. Optimization, as a design aid, is being used in order to develop more efficient devices. An experimental investigation conducted on different ceramic substrates, used as stacks, in standing wave thermo-acoustic coolers shows that geometrical parameters of the stack and the corresponding frequencies are interdependent (Alcock et al., 2017). This is probably the reason behind the growing trend of adopting advanced optimization method in recent studies related to thermo-acoustic refrigerators. A multi-objective optimization of a standing-wave thermo-acoustic refrigerator was carried out by Rao et al. (2017) using the teaching-learning-based optimization algorithm. This study suggest that the stack position, the stack length, the resonator length, the plate thickness and the plate spacing affect the performance of the device. Interestingly, a similar conclusion was reached by Tartibu et al. (2015) using a lexicographic multi-objective optimization approach. Babu and Sherjin (2018) propose the use of Taguchi method for the optimization of a standing-wave thermo-acoustic refrigerator. The stack material, the stack position within the resonator, the frequency and the type of the sound wave were the parameters under investigation. A maximum temperature difference of 5.42^{o}C is reported. The performance of the thermo-acoustic refrigerator based on the temperature difference across the stack was performed by Zolpakar et al. (2016). A Multi-objective Genetic Algorithm (MOGA) optimization scheme was adopted. The optimization of the thermo-acoustic refrigerator stack parameters (namely the stack length and the stack position) using a multi-objective genetic algorithm was done by Zolpakar et al. (2017). An improvement of the COP has been reported, in comparison to previous related studies.

2. Motivation

Previous studies have pointed out the existence of different optimal solutions when optimizing thermo-acoustic refrigerator. It appears that for large scale-devices, the optimal solutions describing the geometrical configuration of the stack is distinct from the ones describing the geometrical configuration of small-scale devices. The COP has been identified as the main parameter to optimize for the former while the cooling load appear to be the main parameter to optimize for the latter (Tartibu et al, 2015 & Herman and Travnicek, 2006). An experimental investigation conducted on a simple standing-wave thermo-acoustic refrigerator has provided clarity on the different between the design for maximum cooling and maximum coefficient of performance in thermo-acoustic refrigerator (Tartibu, 2016). Although the design parameters describing the geometrical configuration of the stack in a standing-wave thermo-acoustic refrigerator are interdependent (Tartibu et al., 2015), the selection of the stack porosity is normally done prior to any investigation, in practice. Therefore, in this paper, two different geometrical configurations of the stack have been analysed namely the stack length and the stack position. In order to measure the performance of the device, the COP is the main objective to optimize. This work aims to provide guidance on the selection of the most suitable geometrical configuration of the stack in order to achieve high performance. In addition, the details of the mathematical programming approach, implemented within the General Algebraic Modelling Systems (GAMS) form part of the contribution of this work. With the GAMS process illustrated in Figure 2, this work provides details of the inputs file and the GAMS compilation models used for the analysis and the optimization of a large scale standing-wave thermo-acoustic refrigerator. A large number of solvers for mathematical programming models have been hooked up to GAMS. LINDOGLOBAL is one of them. The output files consist of Pareto optimal solutions which are the solution of the problem. This is meant to point out to researchers in mathematical analysis and optimization the possibility to formulate and implement their multi-objective problems within the software GAMS.

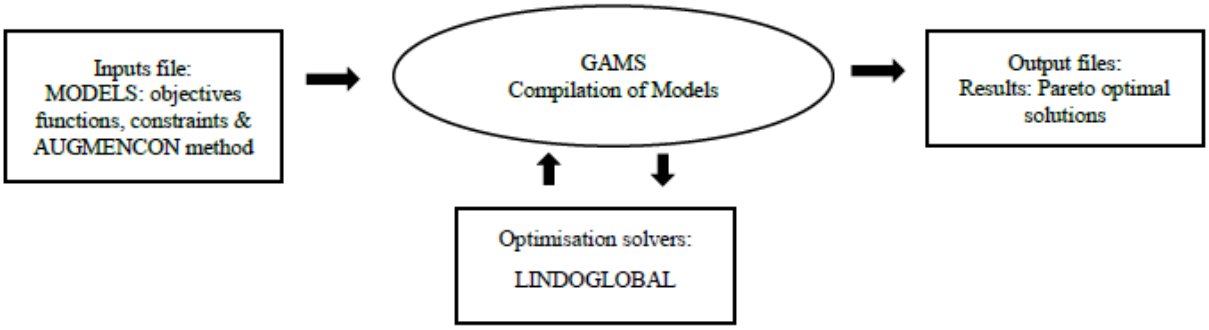


Figure 2. GAMS process illustration

3. Model development

The stack unit constitutes the focus of this study. Some of the parameters normally used to define the stack are shown in Figure 3 namely the stack spacing ($2y_o$), the stack position (X_S) and the plate thickness (2l). The mathematical programming model, proposed in this study, doesn't take the interaction between the performance of the acoustic driver, the heat exchangers effectiveness and the performance of the resonator into account.

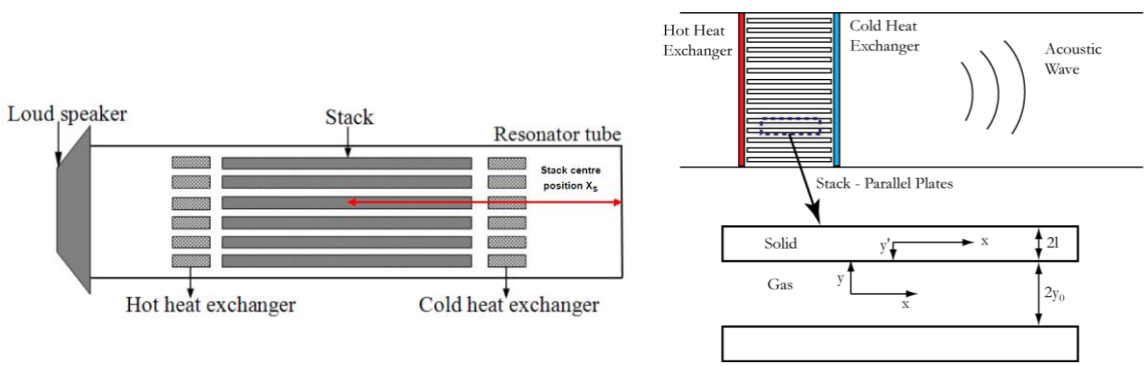


Figure 3. Stack geometrical parameters (Tartibu et al., 2015)

3.1. Stack normalised design parameters

The design of thermo-acoustic refrigerator has to fulfil two main requirements: the achievement of the highest performance while inducing the necessary cooling. One of the major challenge, with the optimization of the device, is the number of independent design parameters that affect the performance of the device. Through normalization, Herman and Travnicek (2006) have collapsed the normalised parameters spaces from nineteen to six. Details of the thermo-acoustic refrigerators parameters are provided in Table 1.

Table 1. Thermo-acoustic refrigerator parameters

	Operation parameters
Drive Ratio (DR)	$DR = \frac{p_0}{p_m}$ Where p_0 and p_m are respectively the dynamic and mean pressure
Normalized temperature difference	$\theta = \Delta T_{mn} = \frac{\Delta T_m}{T_m}$ Where ΔT_m and T_m are respectively the desired temperature span and the mean temperature span
	Gas parameter
Normalized thermal penetration depth	$\delta_{kn} = \frac{\delta_k}{y_0}$ where $2y_0$ is the plate spacing

Stack geometry parameters	
Normalized stack length	$L_{Sn} = \frac{2\pi f}{a} L_S$ where L_S the stack length
Normalized stack position	$X_{Sn} = \frac{2\pi f}{a} X_S$ where f, a and X_S are respectively the resonant frequency, the speed of sound and the stack centre position
Blockage ratio or porosity	$BR = \frac{y_0}{(y_0 + l)}$ where $2l$ is the plate thickness

3.2 Objectives functions

The derivations of objectives functions considered in this study are available in Tijani (2001). The normalized heat flow Φ_H and acoustic power Φ_W are the main objective functions to optimize. They are given by the following equations:

$$\Phi_H = -\left[\frac{\delta_{kn} DR^2 \sin(2X_{Sn})}{8\gamma(1+\sigma)\left(1-\sqrt{\sigma}\delta_{kn} + \frac{1}{2}\sigma\delta_{kn}^2\right)}\right] \times \left[\frac{\Delta T_{mn} \tan(X_{Sn})}{(\gamma-1) BR\ L_{Sn}} \times \frac{\left(1+\sqrt{\sigma}+\sigma\right)}{1+\sqrt{\sigma}} - \left(1+\sqrt{\sigma}-\sqrt{\sigma}\delta_{kn}\right)\right] \tag{1}$$

$$\Phi_W = \left[\frac{\delta_{kn} DR^2 L_{Sn} (\gamma-1) BR\ \cos^2(X_{Sn})}{4\gamma}\right] \times \left[\frac{\Delta T_{mn} \tan(X_{Sn})}{BR\ L_{Sn}(\gamma-1)\left(1+\sqrt{\sigma}\right)\left(1-\sqrt{\sigma}\delta_{kn} + \frac{1}{2}\sigma\delta_{kn}^2\right)} - 1\right]$$
$$-\left[\frac{\delta_{kn} L_{Sn} DR^2}{4\gamma} \times \frac{\sqrt{\sigma}\sin^2(X_{Sn})}{BR\left(1-\sqrt{\sigma}\delta_{kn} + \frac{1}{2}\sigma\delta_{kn}^2\right)}\right] \tag{2}$$

The normalized cooling load Φ_C and the coefficient of performance COP of the stack are obtained respectively as follows:

$$\Phi_C = \Phi_H - \Phi_W \tag{3}$$

$$COP = \frac{\Phi_H - \Phi_W}{\Phi_W} \tag{4}$$

The cooling load Φ_C is dependent on the following 8 non-dimensional parameters:

$$\Phi_C = F\left(\sigma,\ \gamma,\ \varepsilon_S,\ T_{mn},\ L_{Sn},\ X_{Sn},\ BR,\ \delta_{kn}\right) \tag{5}$$

Where σ, γ, ε_S and T_{mn} represent respectively the Prandtl number, the polytropic coefficient, the stack heat capacity correction factor and the normalized temperature difference. Considering the boundary layer approximation, adopted in most thermo-acoustic refrigerator modelling, the acoustic power loss per unit area of the resonator is given as follows:

$$\overset{o}{W}_2 = \frac{dW_2}{dS} = \left[\frac{\delta_{kn} DR^2 L_{Sn} (\gamma-1) BR\ \cos^2(X_{Sn})}{4\gamma}\right] + \left[\frac{\delta_{kn} L_{Sn} DR^2}{4\gamma} \times \frac{\sqrt{\sigma}\sin^2(X_{Sn})}{BR\left(1-\sqrt{\sigma}\delta_{kn} + \frac{1}{2}\sigma\delta_{kn}^2\right)}\right] \tag{6}$$

4. Mathematical programming formulation

The standing-wave thermo-acoustic stack performance has been formulated as a multi-objective mathematical programming problem. Three objectives functions namely the cooling load (Φ_C), the coefficient of performance (COP) and the acoustic power lost ($\overset{o}{W}_2$) are considered as the three distinct objective components to optimize. A single objective optimization has been performed for each of these objective. Detailed descriptions of this approach are available in Tartibu et al. (2015). This study demonstrates that the three objectives functions are conflicting making the problem described in this study suitable for a multi-objective approach. Considering that one of the endogenous model is discontinuous, the problem was impossible to solve as a non-linear problem. Hence, the optimisation task has been formulated as a three-criterion non-linear programming problem with discontinuous derivatives (DNLP). This optimisation task aims to maximise the magnitude of the cooling load (Φ_C), maximise the coefficient of performance (COP) and minimise acoustic power lost ($\overset{o}{W}_2$).

$$(\mathrm{MPF}) \max_{L_{Sn}, X_{Sn}} \xi = \left\{ \Phi_{C(L_{Sn}, X_{Sn})}, COP_{(L_{Sn}, X_{Sn})}, -\overset{o}{X}_{2(L_{Sn}, X_{Sn})} \right\} \tag{7}$$

subject to bound limits $\Phi_{C\max}$, $\Phi_{C\min}$, COP_C (the details about the computation of these bound limits are available in Tartibu et al., 2015). Since a negative cooling load doesn't have any physical meaning, the following constraint has been enforced:

$$\Phi_C = \Phi_H - \Phi_W > 0 \tag{8}$$

The subscripts (L_{Sn}, X_{Sn}) in (7) denotes the parameters of the standing wave thermo-acoustic refrigerator. The outcome of the optimization of multiple objectives function is a multitude of optimal solutions. As a designer, the most preferred solutions are generally sought.

Therefore, the lexicographic optimisation approach has been adopted in this study in order to construct the payoff table and yield "Pareto optimal solutions" only while avoiding weak (or non-efficient) solutions. This new mathematical programming approach (referred to as augmented ε-constraint method or AUGMENCON) was developed by Mavrotas (2009). The advantages and the implementation of this approach within the software GAMS (General Algebraic Modelling System) can be found in Mavrotas (2009).

The code developed in order to solve the multi-objective optimization problem related to the standing-wave thermo-acoustic refrigerator has been disclosed in the Appendix. It should be noted that the part of the code that performs the calculation of payoff table with lexicographic optimisation and the generation of the optimal solutions are readily available in the GAMS library (http://www.gams.com/modlib/libhtml/epscm.htm).

The formulation of the problem is critical to the success of this optimization endeavour. Details related to the application of the augmented ε-constraint method, the selection of the most important function (or primary objective function) in the formulation of the problem could be found in Tartibu et al. (2015) and are beyond the scope of the work presented here. Subsequently, the augmented ε -constraint method for solving the model (7) can be formulated as follows:

$$\max\left(COP_{(L_{Sn}, X_{Sn}, BR, \delta_{kn})} + dir_1\, r_1 \times \left(\frac{s_2}{r_2} + \frac{s_3}{r_3} + \frac{s_4}{r_4} + \frac{s_5}{r_5} \right) \right) \tag{9}$$

Subject to. $\Phi_{C(L_{Sn}, X_{Sn}, BR, \delta_{kn})} - dir_2\, s_2 = \varepsilon_2$

$$\overset{o}{W}_{2(L_{Sn}, X_{Sn}, BR, \delta_{kn})} - dir_3\, s_3 = \varepsilon_3$$

$s_i \in \Re^+$

where dir_i is the direction of the ith objective function (equal to -1 when the ith function should be minimised, and +1 when it should be maximised). Through parametrical iterative variations ε_i, efficient solutions to the problem could be obtained by parametrical iterative variations in the ε_i. s_i and $r_1\, s_i / r_i$ represent the surplus variables for the

constraints and the second term of the objective functions used to avoid any scaling issue, respectively. The following variables constraints (upper and lower bounds) was considered during the optimization:

$X_{Sn}.lo = 0.010;\ X_{Sn}.up = 1.000$

The proposed DNLP problem has been solved by GAMS 23.8, using LINDOGLOBAL solver on a personal computer Pentium IV, 2.2 GHz with 4 GB RAM. The normalised stack length (L_{Sn}) has been arbitrarily given successive values of 0.2-0.3-0.4-0.5. The optimal solutions corresponding to each value of L_{Sn} have been generated. The maximum CPU time taken to complete the results is approximatively 10 sec.

5. Results and discussions

In order to study the results trends and establish the relationship between the stack geometrical parameters, three different case study, related to different stack porosity, have been considered:

- Case 1: Blockage ratio (BR) = 0.7 & normalised thermal penetration depth (thermp) = 0.046;
- Case 2: Blockage ratio (BR) = 0.8 & normalised thermal penetration depth (thermp) = 0.069;
- Case 3: Blockage ratio (BR) = 0.9 & normalised thermal penetration depth (thermp) = 0.092;

The results obtained have been reported in Table 2. From this results, it appears clearly that there is a specific stack length corresponding to a specific stack position in order to achieve maximum performance of the device. These results makes it easy for a decision maker to make a selection of suitable geometrical configuration based on the design requirement. The highest CPU time was 5.250 sec making this process efficient with respect to the running time cost.

Table 2. Optimal solutions computed using AUGMENCON

	Lsn	Xsn	COP	CPU time (sec)
BR = 0.7 & thermp = 0.046	0.1	0.426	16.01%	5.250
	0.2	0.479	7.41%	2.432
	0.3	0.499	4.16%	1.363
	0.4	0.509	2.44%	0.801
	0.5	0.515	1.39%	0.455
BR = 0.8 & thermp = 0.069	0.1	0.484	14.56%	4.775
	0.2	0.531	7.41%	2.430
	0.3	0.552	4.15%	1.360
	0.4	0.563	2.44%	0.799
	0.5	0.569	1.55%	0.508
BR = 0.9 & thermp = 0.092	0.1	0.520	15.99%	5.244
	0.2	0.580	7.40%	2.426
	0.3	0.602	4.14%	1.358
	0.4	0.613	2.43%	0.797
	0.5	0.619	1.55%	0.510

The results obtained have been presented graphically in Figure 4(a), (b) and (c). With respect to the coefficient of performance, the trends observed suggest that irrespective of the porosity of the stack considered and in order to achieve high performance of the device:

- decreasing the stack length is beneficial to the performance of the device;
- the maximum performance is expected with shorter stacks.

With respect to the stack position and the stack length and in order to achieve high performance for the device:

- shorter stack must be positioned nearer the closed end of the resonator tube and
- longer stack must be positioned relatively far away from the closed end of the resonator;
- while the trends reported in Figure 4(a) and (c) shows that the influence of the stack positioning is smaller, Figure 4(b) indicate a significant change of the stack positioning with respect to the stack length. This could be attributed to the non-linearity reported in previous studies suggesting that design parameters describing the stack are somehow interdependent.

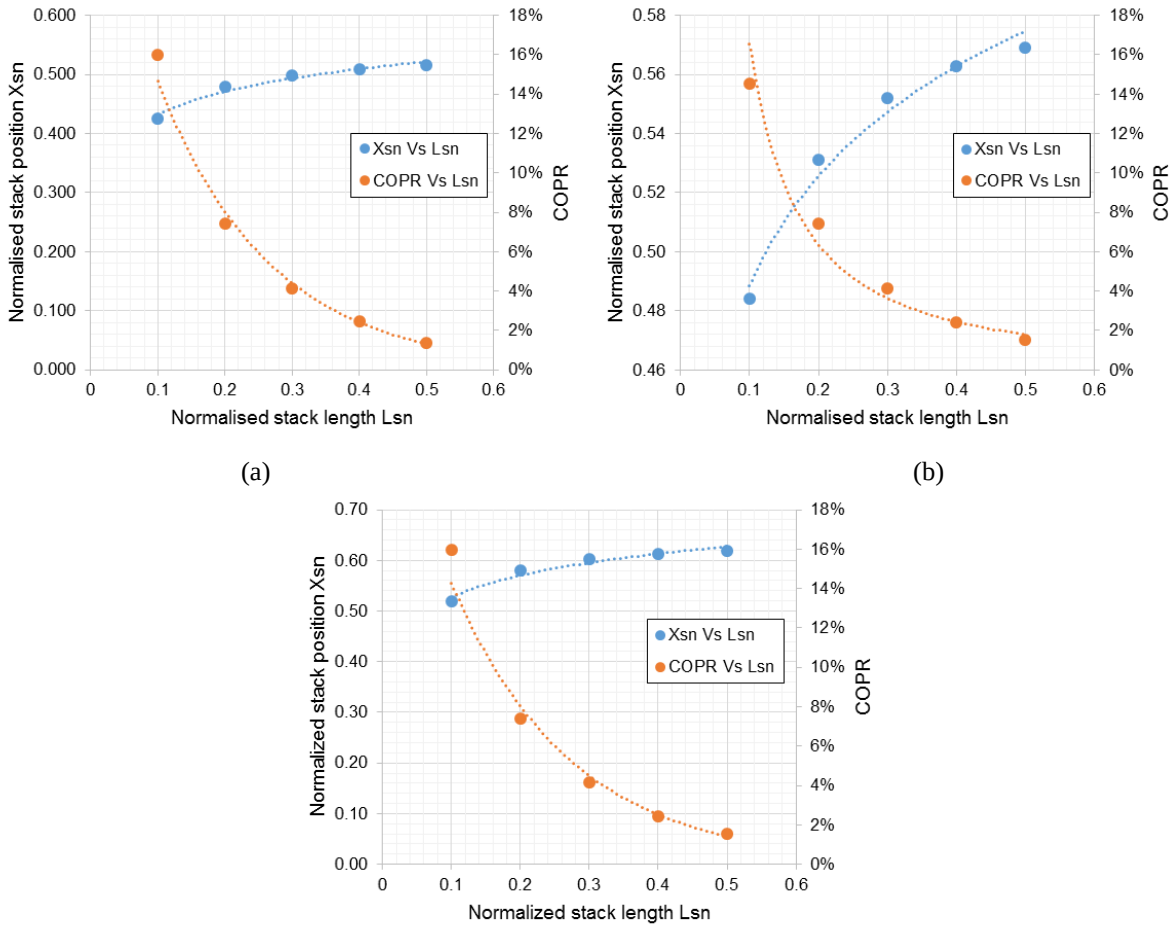


Figure 4. Normalised stack position and COPR as a function of the normalised stack length corresponding to the porosity (a) BR = 0.7 & thermp = 0.046; (b) BR = 0.8 & thermp = 0.069 and (c) BR = 0.9 & thermp = 0.092

Conclusion

The designing of thermo-acoustic refrigerator offers significant challenges because of the multitude of design parameters and the interdependence between these parameters. In this paper, the design and optimization of the stack of a thermo-acoustic refrigerator is described. A mathematical programming model has been developed in order to compute optimal solutions that correspond to the best geometrical configuration with respect to the COP. The developed model has been implemented within the software GAMS. A randomly selected sets of porosity have been investigated with the aim of locating the most suitable stack length and position as this is generally the case in practice. The analysis of the results obtained reveals that the best COP is achieved with relatively shorter stack. With regards to the geometry of the device and irrespective of the porosity, shorter stack must be positioned nearer the closed end of the resonator tube while the opposite is true for longer stack. The proposed mathematical programming model gives a fast engineering estimate of the geometry of the stack allowing the decision maker to make an informed selection.

References

Alcock, A.C., Tartibu, L.K. and Jen, T.C. (2017). *Experimental investigation of ceramic substrates in standing wave thermoacoustic refrigerator*. Procedia Manufacturing, 7, 79-85.

Babu, K.A. and Sherjin, P. (2018). *Experimental investigations of the performance of a thermoacoustic refrigerator based on the Taguchi method*. Journal of Mechanical Science and Technology, 32(2), 929-935.

Ciconkov, R., and Zahid, H. A. (2009). *New Technologies in Ammonia Refrigerating and Air-Conditioning Systems*, Heat Transfer Engineering, 30(4), 324–329.

Generalized Algebraic modelling Systems, (GAMS), [online]. http://www.gams.com

Herman, C. and Travnicek, Z. (2006). *Cool sound: the future of refrigeration? Thermodynamic and heat transfer issues in thermoacoustic refrigeration*. Heat and mass transfer, 42(6), 492-500.

Mavrotas, G. (2009). *Effective implementation of the ε-constraint method in multi-objective mathematical programming problems*. Applied mathematics and computation, 213(2), 455-465.

Rao, R.V., More, K.C., Taler, J. and Ocłoń, P. (2017). *Multi-objective optimization of thermo-acoustic devices using teaching-learning-based optimization algorithm*. Science and Technology for the Built Environment, 23(8), 1244-1252.

Tartibu, L.K. (2016). *Maximum cooling and maximum efficiency of thermoacoustic refrigerators*. Heat and Mass Transfer, 52(1), 95-102.

Tartibu, L.K., Sun, B. and Kaunda, M.A.E. (2015). *Lexicographic multi-objective optimization of thermoacoustic refrigerator's stack*. Heat and Mass Transfer, 51(5), 649-660.

Tijani M.E.H. (2001). *Loudspeaker-driven thermo-acoustic refrigeration*, Ph.D. thesis, Eindhoven University of Technology, Netherlands.

Zolpakar, N.A., Mohd-Ghazali, N. and Ahmad, R. (2016). *Experimental investigations of the performance of a standing wave thermoacoustic refrigerator based on multi-objective genetic algorithm optimized parameters*. Applied Thermal Engineering, 100, 296-303.

Zolpakar, N.A., Mohd-Ghazali, N. and Ahmad, R. (2017). *Optimization of the Stack Unit in a Thermoacoustic Refrigerator*. Heat Transfer Engineering, 38(4), 431-437.

Appendix

```
$ Title Thermoacoustic refrigerator
scalars
* Inputs parameters
         isco isentropic coefficient /1.63/
         pn prandtl number /0.67/
         deltat temperature difference /0.030/
         tw wall thickness /0.012/
         DR drive ratio /0.035/
         BR /0.9/
         thermp /0.092/
         Lsn /0.2/
         ; scalar starttime;
         starttime=jnow;
Sets
k objective functions / cop, coolingload,acousticpowerlost/;
$ set min -1
$ set max +1
Parameter dir(k) direction of the objective functions / cop %max%,
coolingload %max%,acousticpowerlost %min%/;
Positive VARIABLES  Xsn, eqA, eqB, eqC, eqD, eqE, eqF, eqG, eqH, eqJ;
Xsn.lo=0.01; Xsn.up=1;
Free VARIABLE
z(k) objective function variables;
Equations
objcop objective for maximising cop
objcoolingload objective for maximising coolingload
objacousticpowerlost objective for minimising acousticpowerlost
```

```
eqAdef equation
eqBdef equation
eqCdef equation
eqDdef equation
eqEdef equation
eqFdef equation
eqGdef equation
eqHdef equation
eqJdef equation
ebound1 equation;
* Equations used to simplify the objective functions equations
eqAdef.. eqA =e= 2*Xsn;
eqBdef.. eqB =e= thermp*(DR**2)*sin(eqA);
eqCdef.. eqC =e= 0.5*pn*SQR(thermp);
eqDdef.. eqD =e= SQRT(pn)*thermp;
eqEdef.. eqE =e= tan(Xsn);
eqFdef.. eqF =e= (isco-1)*BR*Lsn;
eqGdef.. eqG =e= 1+SQRT(pn)-eqD;
eqHdef.. eqH =e= cos(Xsn)*cos(Xsn);
eqJdef.. eqJ =e= sin(Xsn)*sin(Xsn);
* Design constraints
ebound1.. abs((((deltat*eqE*(1+SQRT(pn)+pn))/(eqF*(1+SQRT(pn))))-
eqG)/((8*isco*(1+pn)*(1-eqD+eqC))/eqB)) =g=
abs((((thermp*eqF*(DR**2)*eqH)/(4*isco)))*(((deltat*eqE)/(eqF*(1+SQRT(pn))*(1
-eqD+eqC)))-1)-((eqD*Lsn*(DR**2)*eqJ)/(4*isco*BR*(1-eqD+eqC))));
objcop..
* Objective functions
z('cop')*abs(((((thermp*eqF*(DR**2)*eqH)/(4*isco)))*(((deltat*eqE)/(eqF*(1+SQ
RT(pn))*(1-eqD+eqC)))-1)-((eqD*Lsn*(DR**2)*eqJ)/(4*isco*BR*(1-eqD+eqC)))))
=e= abs((((deltat*eqE*(1+SQRT(pn)+pn))/(eqF*(1+SQRT(pn))))-
eqG)/((8*isco*(1+pn)*(1-eqD+eqC))/eqB))-
(abs(((((thermp*eqF*(DR**2)*eqH)/(4*isco)))*(((deltat*eqE)/(eqF*(1+SQRT(pn))*
(1-eqD+eqC)))-1)-((eqD*Lsn*(DR**2)*eqJ)/(4*isco*BR*(1-eqD+eqC))))));
objcoolingload.. z('coolingload') =l=
abs((((deltat*eqE*(1+SQRT(pn)+pn))/(eqF*(1+SQRT(pn))))-
eqG)/((8*isco*(1+pn)*(1-eqD+eqC))/eqB))-
(abs(((((thermp*eqF*(DR**2)*eqH)/(4*isco)))*(((deltat*eqE)/(eqF*(1+SQRT(pn))*
(1-eqD+eqC)))-1)-((eqD*Lsn*(DR**2)*eqJ)/(4*isco*BR*(1-eqD+eqC))))));
objacousticpowerlost..  z('acousticpowerlost') =e=
((thermp*eqF*(DR**2)*eqH)/(4*isco))+((eqD*Lsn*(DR**2)*eqJ)/(4*isco*BR*(1-
eqD+eqC)));
* Bound limits as per Ref (Tartibu et al., 2015)
z.lo('cop') = 0.01; z.up('cop') = 32.8;
z.lo('coolingload') = 4.3339E-8; z.up('coolingload') = 7.2659E-4;
* Extra bound limits in order to solve division by zero errors
eqB.lo = 0.000001; eqB.up = 0.001;
eqF.lo = 0.0001; eqF.up = 0.5;
MODEL thermoacoustic /ALL/;
Option Reslim=1000000;

$STitle eps-constraint method
Set k1(k) the first element of k, km1(k) all but the first elements of k;
k1(k)$(ord(k)=1) = yes; km1(k)=yes; km1(k1) = no;
Set kk(k) active objective function in constraint allobj
Parameter rhs(k) right hand side of the constrained obj functions in eps-
constraint
maxobj(k) maximum value from the payoff table
```

```
minobj(k) minimum value from the payoff table
Variables a_objval auxiliary variable for the objective function
obj auxiliary variable during the construction of the payoff table
Positive Variables sl(k) slack or surplus variables for the eps-constraints
Equations
con_obj(k) constrained objective functions
augm_obj augmented objective function to avoid weakly efficient solutions
allobj all the objective functions in one expression;
con_obj(km1).. z(km1) - dir(km1)*sl(km1) =e= rhs(km1);
* The first objective function is optimised and the others are used as
constraints
* The second term is added to avoid weakly efficient points
augm_obj.. sum(k1,dir(k1)*z(k1))+1e-3*sum(km1,sl(km1)/(maxobj(km1)-
minobj(km1))) =e= a_objval;
allobj.. sum(kk, dir(kk)*z(kk)) =e= obj;
Model mod_payoff / thermoacoustic, allobj / ;
Model mod_epsmethod / thermoacoustic, con_obj, augm_obj / ;
option limrow=10, limcol=10;
option solprint=on, solvelink=%solvelink.CallModule%;
option decimals=8;
Parameter payoff(k,k) payoff tables entries;
Alias(k,kp);
* Payoff table is generated by applying lexicographic optimisation
loop(kp, kk(kp)= yes;
repeat solve mod_payoff using dnlp maximising obj;
payoff(kp,kk) = z.l(kk);
z.fx(kk) = z.l(kk);
*// The value of the last objective optimised are freezed
kk(k++1) = kk(k);
*// cycle through the objective functions
until kk(kp); kk(kp) = no;
* Values of the objective functions are released for the new iteration
z.up(k) = inf; z.lo(k) =-inf; );
if (mod_payoff.modelstat<>%modelstat.Optimal%, abort 'no optimal solution for
mod_payoff');
display payoff;
minobj(k)=smin(kp,payoff(kp,k));
maxobj(k)=smax(kp,payoff(kp,k));
$set fname p.%gams.scrext%
File fx solution points from eps-method / "%gams.scrdir%%fname%" /;
$if not set gridpoints $set gridpoints 5
Set g grid points /g0*g%gridpoints%/
grid(k,g) grid
Parameter
gridrhs(k,g) rhs of eps-constraint at grid point
maxg(k) maximum point in grid for objective
posg(k) grid position of objective
firstOffMax, lastZero some counters
numk(k) ordinal value of k starting with 1
numg(g) ordinal value of g starting with 0;
lastZero=1; loop(km1, numk(km1)=lastZero; lastZero=lastZero+1); numg(g) =
ord(g)-1;
grid(km1,g) = yes;
*// Define the grid intervals for different objectives
maxg(km1) = smax(grid(km1,g), numg(g));
gridrhs(grid(km1,g))$(%min%=dir(km1)) = maxobj(km1) -
numg(g)/maxg(km1)*(maxobj(km1)- minobj(km1));
```

```
gridrhs(grid(km1,g))$(%max%=dir(km1)) = minobj(km1) +
numg(g)/maxg(km1)*(maxobj(km1)- minobj(km1));
display gridrhs;
* Walking through the grid points and taking shortcuts if the model becomes
infeasible
posg(km1) = 0;
repeat
rhs(km1) = sum(grid(km1,g)$(numg(g)=posg(km1)), gridrhs(km1,g));
solve mod_epsmethod maximising a_objval using dnlp;
if (mod_epsmethod.modelstat<>%modelstat.Optimal%,
*// not optimal is in this case infeasible
lastZero = 0; loop(km1$(posg(km1)>0 and lastZero=0), lastZero=numk(km1));
posg(km1)$(numk(km1)<=lastZero) = maxg(km1);
*// skip all solves for more demanding values of rhs(km1)
else
loop(k, put fx z.l(k):12:2); put /);
* Proceed forward in the grid
firstOffMax=0;
loop(km1$(posg(km1)<maxg(km1) and firstOffMax=0),posg(km1)=posg(km1)+1;
firstOffMax=numk(km1));
posg(km1)$(numk(km1)<firstOffMax) = 0;
until sum(km1$(posg(km1)=maxg(km1)),1)=card(km1) and firstOffMax=0;
putclose fx;
*// close the point file
```

Biography

Lagouge Tartibu is a senior lecturer in the department of Mechanical Engineering Technology at the University of Johannesburg in South Africa. He has been a Lecturer for Cape Peninsula University of Technology (2007-2012) and Mangosuthu University of Technology (2013-2015). He holds a Doctorate degree in Mechanical engineering from Cape Peninsula University of Technology (2014) and a Bachelor degree in Electromechanical Engineering from the University of Lubumbashi (2006). His primary research areas are thermal science, electricity generation and refrigeration using thermo-acoustic technology, mathematical analysis/optimization and mechanical vibration.