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On Minimum Cost Non-Uniform Sampling Schemes for Optimal Design of Control Charts: Application to $\bar{X}$ and T^2 Control Charts

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Abstract

As an economically superior approach to uniform sampling scheme, the pioneering work of Rahim and Banerjee on sampling procedure with non-uniform intervals has been broadly employed in statistical process monitoring during the past three decades. However, since its consecutive times of inspection have to be determined through a function of the first sampling interval, it might make some kind of complexity in practical administration compare to uniform approach. In this paper, the intuitive companionship between the sampling frequencies and the failure rate of a process is discussed by investigating various functional constraints on choosing the length of sampling intervals and their effects on the optimal design of control charts, specifically for processes which deteriorate over time. Extensive numerical illustrations are prepared for monitoring univariate and multivariate quality characteristics in manufacturing and service sectors following an increasing failure rate Weibull shock model. The results obtained from proposed structures for sampling intervals could slightly improve the expected cost per unit time which illustrate Rahim-Banerjee model inducing constant integrated hazard over each sampling interval is a well-established near-optimal non-uniform approach. However, further investigation on the mathematical problem of finding the sampling scheme by which the cost function is optimized would be highly fruitful.

Keywords
Process Failure Mechanism, Integrated Hazard over Sampling Interval, Quality Characteristic(s) Distribution, Inspection Procedures, and Non-linear Optimization Problem.

1. Introduction

As a permanent feature of the increasingly competitive international business environment, quality control of a production or service process would be modeled mathematically by monitoring the shift(s) in parameter(s) of the underlying Quality Characteristic Distribution (QCD). Control chart, the pioneering work of Shewhart (1924) in the economic control of variation for statistical stability and nowadays the heart of Statistical Process Control (SPC), is a graphical way of inspecting a process by consecutive sample points in the order in which they were obtained in time and signaling the occurrence of assignable cause of variation if an unusual shift in process has occurred. The detection and elimination of the assignable cause will bring the process back to control.

To design a control chart, the quality practitioner should specify three categories of decision variables, called as design parameters, namely the sample size, the sampling time interval (e.g., h in Uniform Sampling Scheme (USS) or h_j for $j = 1, 2, ...,$ in Non-uniform Sampling Scheme (NSS)) and the control limits of the chart which would construct the critical region of its corresponding hypotheses testing (e.g., $\mu \mp L\frac{\sigma}{\sqrt{n}}$ for $\bar{X}$ control chart, when the target QCD's parameters μ and σ assumed known or estimated). Observing sample averages $\bar{X}$ outside the control limits is considered as a (probable) signal of the out-of-control process.

There are mutual relations between the design parameters and the control chart's statistical criteria like Type-I and Type-II error probabilities which can be considered as performance measures, respectively for the rate of false alarms and for correctly identifying a shift in the process when one exists. On the other hand, different design parameters require diverse budgeting. A series of research studies in the literature has been devoted to control chart's Economic Design (ED), traced back first to Duncan (1956) and unified for any chart, control statistic, or QCD by Lorenzen and Vance (1986), whose optimal design parameters minimize an objective function, the Expected total Cost per unit Time

(ECT) or the cost model. For reviews on ED, see Gibra (1975), Montgomery (1980), Vance (1983), Svoboda (1991), and Ho and Case (1994). The minimization model subject to statistical restrictions on Type-I error or false alarm rate, the power or capability to detect undesirable shifts, etc. is called Economic Statistical Design (ESD) first proposed by Saniga (1989). For more details, see also Zhang and Berardi (1997), Al-Oraini and Rahim (2003), Chen and Cheng (2007), Chen and Yeh (2011), etc.

Mathematical modeling of these design approaches depends on two underlying distributions: the QCD for computing Type-I and Type-II error probabilities, and the Process Failure Mechanism (PFM) for studying the in-control time of the process. Lorenzen and Vance (1986) assumed that the PFM or the system's shock model follows an exponential distribution which has identified uniquely by memoryless property and constant failure rate. There have been also many processes with decreasing, bathtub, and increasing failure rate (IFR) for which the non-Markovian shock models are relevant. Hu (1986) and McWilliams (1989) showed that the USS-based ECT is insensitive to choosing shock models if their mean values are the same, and thus the unified model of Lorenzen and Vance (1986), beside the arbitrary QCD, could be used for any control chart regardless of the system's shape of the failure rate function. Often, the process deterioration starts gradually due to the ageing effects of wear, corrosion, fatigue and other related phenomena. Process deterioration is also experienced due to overheating and vibration, which are commonly encountered in many industries, including metal cutting, etching, plating, moulding, and so forth (Montgomery 1992).

Since the hazard rate of the exponential distribution is independent of time, the USS-based Integrated Hazard over each sampling Interval (IHI) and the Shift Probability on each sampling Interval given that the process is in control at the first of that interval (SPI) will be CONSTANT for Markovian processes. Motivating by this and using a recursive equations approach, an alternative approach to cycle partition method for finding the ECT, Banerjee and Rahim (1988) achieved a NSS-based cost model by applying the constant IHI, or equivalently, the constant SPI for systems with IFR Weibull PFM as a constraint on sampling intervals. The proposed NSS results in a lower cost than USS which induces increasing IHI/SPI for IFR processes, and so NSS is superior to the USS in an economic sense. In addition, they illustrated that the shock model mis-specification has a significant impact on the improved NSS-based ECT, and so it should be accurately estimated based on the available knowledge from the process history. Sensitivity analyses in literature of the ESD have illustrated that the shock model parameters' effects are also significant on the outgoing quantities of the optimization problem.

Dating from the pioneering work of Banerjee and Rahim (1988), many researchers have improved various SPC methods using the proposed NSS approach. Maintaining IFR Weibull PFM, Their work has been extended to dynamic ED by Ohta and Rahim (1997), to integrated production maintenance and quality model for imperfect processes by Ben-Daya (1999), to joint $\bar{X} - R$ charts by Rahim and Costa (2000), to systems with multiple assignable causes by Chen and Yang (2002a), to non-Shewhart CCs by Chen and Yang (2002b), to ESD for multivariate QCD by Yang and Rahim (2005), to ED under non-normal QCD by Chen and Yeh (2011), to ED integrated with Taguchi loss function by pasha et al. (2018), etc. In all of the mentioned models, the proposed NSS dominates the USS. On the other hand, Rahim and Banerjee (1993) (hereafter RB) generalized their NSS-based economic model which utilizes constant IHI/SPI approach to an arbitrary shock model. The reader may refer to Pasha et al. (2017a) for a comprehensive discussion on generalization of the optimal design of control charts in terms of QCD, control statistic (joint, multivariate, etc.), sampling schemes, and PFM.

Pasha et al. (2017b) proposed two instances of decreasing IHI/SPI approach as conservative functional constraints which would be used for determining inspection intervals in ED and ESD of highly sensitive IFR processes. Similar to RB, their model implies that jth sampling interval is computed through a decreasing function of j which depends only on first interval h_1. In comparison to constant approach of RB, they showed the ECT values increase if one desire to have the conservative decreasing trends for IHI/SPI.

In this paper, we impose various non-uniform functional constraints directly on the sampling intervals by which the length of intervals decreases over time. The scheme with decreasing sampling interval is a result of applying the constant and decreasing IHI/SPI for IFR systems, however, there is no condition on IHI/SPI in this paper. In other words, IHI/SPI are not necessarily decreasing or even constant as the system gets old. The purpose is to compare different non-uniform sampling schemes (decreasing sampling intervals with even non-monotone IHI/SPI) for optimal design of control chart of deteriorating processes in terms of their corresponding ECTs, specifically to compare them with RB (decreasing sampling intervals with constant IHI/SPI) and USS (constant sampling intervals with increasing

IHI/SPI). We present a comprehensive numerical investigation of the effects of these structures on the $\bar{X}$ control chart in the case of univariate QCD and on the T^2 Hotelling control chart in the case of multivariate QCD.

The rest of the article is as follows. Section (2) presents the generalized ECT which can be adapted for any QCD and PFM. In addition, four decreasing procedures of sampling intervals are discussed to use for the numerical studies. Section (3) illustrates numerically how the different procedures of sampling intervals effect on the optimal design parameters and the other output quantities of the economic design. A conclusion section completes the article.

2. The Cost Model

A process can be divided into quality cycles which begins with the process in the in-control state and continues until a true out-of-control signal appears on the control chart. Following an adjustment in which the process returns to the in-control state, a new cycle begins (See Figure 1). Therefore, by defining a quality cycle as the time between the start of successive in-control periods and applying the renewal reward theorem (Ross 1970), the cost model ECT can be obtained by the ratio of the expected cost of the cycle $E(C)$ to its expected time $E(T)$. Beside some process time and cost parameters, mathematical modeling of these quantities depends on the QCD for computing Type-I and Type-II error probabilities, α and β respectively, and the shock model.

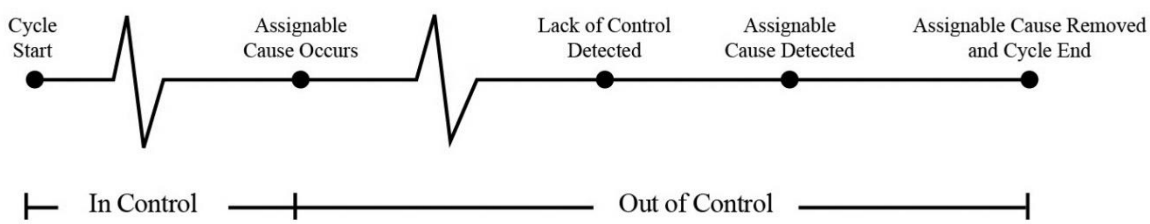

Figure 1. The quality cycle

2.1 Basic assumptions and notations

RB presented a generalized cost model which can be adapted for any PFM (lifetime random variable T_a with distribution function F, density function f, failure rate function r, and expectation $E(T_a)$), QCD, and sampling scheme. The process is monitored by selecting successive random samples each of size n at time point $t_i = \sum_{j=1}^{n} h_j$, where h_j denotes the jth sampling interval for $j = 1, 2, ...$ (For the case of USS, $h_j = h$ and $t_i = ih$). In addition, p_j (SPI) is defined as the conditional probability that the process shifts to the out-of-control state during the time interval (t_{j-1}, t_j) given that the process was at the in-control state at time t_{j-1}; i.e., $p_j = \frac{\int_{t_{j-1}}^{t_j} f(t)dt}{\int_{t_{j-1}}^{\infty} f(t)dt} = \frac{F(t_j)-F(t_{j-1})}{1--F(t_{j-1})}$. Define $p_0 = 0$ and $t_0 = 0$. RB proved the following expressions are satisfied for the expected time and expected cost of a quality cycle, respectively:

$$ET = \sum_{j=1}^{\infty} h_j \prod_{i=0}^{j-1}(1-p_i) + \alpha Z_0 \sum_{j=1}^{\infty}\prod_{i=1}^{j}(1-p_i) + \beta \sum_{j=1}^{\infty} p_j \prod_{i=0}^{j-1}(1-p_i) \sum_{i=j+1}^{\infty} h_i\, \beta^{i-j-1} + Z_1, \qquad (1)$$

and

$$EC = D_0 \sum_{j=1}^{\infty} h_j \prod_{i=0}^{j-1}(1-p_i) + \alpha Y \sum_{j=1}^{\infty}\prod_{i=1}^{j}(1-p_i) + (D_0 - D_1)\tau + (D_1 - D_0)\sum_{j=1}^{\infty} \omega_j\, p_j \prod_{i=0}^{j-1}(1-p_i)$$

$$+ D_1 \beta \sum_{j=1}^{\infty} p_j \prod_{i=0}^{j-1}(1-p_i) \sum_{i=j+1}^{\infty} h_i\, \beta^{i-j-1} + (a+bn)\left[\frac{\beta}{1-\beta} + \sum_{j=0}^{\infty}\prod_{i=1}^{j}(1-p_i)\right] + W, \qquad (2)$$

where Z_0, Z_1, a, b, D_0, D_1, Y, and W are process parameters introduced in Table 1.

Table 1. The process cost and time parameters and the design parameters

Category of Variables	**Notation**	**Expression**
Design Parameters	n	Sample size
	$h\ (h_j)$	Time interval(s) between successive samples
	L	Number of standard deviations from control limits to center line
Time Parameters	Z_0	Expected assignable cause search time for a false alarm
	Z_1	Expected time to identify the assignable cause and repair the process
Cost Parameters	a	Fixed cost per sample
	b	Variable cost per sample
	D_0	Hourly cost due to nonconformities produced while in control
	D_1	Hourly cost due to nonconformities produced while out of control
	Y	Cost per false alarm
	W	Cost for locating and repairing the assignable cause

As $\Pr(T_a > t_{j-1}) = 1 - F(t_{j-1}) = \bar{F}(t_{j-1}) = \prod_{i=1}^{j-1}(1-p_i)$ and $\Pr(t_{j-1} < T_a < t_j) = F(t_j) - F(t_{j-1}) = \prod_{i=1}^{j-1}(1-p_i)$, the time and cost expectations of (1) and (2) can be equivalently written as expressions based on F. These generalized models would be adapted for a process in terms of both quality data (QCD and the control statistic) and failure data (PFM). Due to the renewal reward theorem, one can write ECT=$\frac{E(C)}{E(T)}$.

In ED, we are interested in finding the optimal design parameters n, h_j, and L that minimize the nonlinear function ECT for given values of time, cost, shift and distribution parameters. In ESD, the objective function ECT would be minimized subject to some constraints on statistical criteria, e.g. the probabilities of type I and type II errors: $\alpha \le \alpha_U$ and $\beta \le \beta_U$, where α_U and β_U are specified values. By replacing input cost and time parameters of the process, and then by considering h_j and its corresponding p_j in terms of h_1, the ECT will be a non-linear objective function of three categories of design parameters. For any triple vector of design parameters, the value of ECT can be obtained by setting convergence criteria for the series expressions in the model.

2.2 Discussion on Sampling Intervals

To obtain the optimal design parameters in NSS-based optimization problem, one needs to simplify the model in terms of the length of sampling intervals, as this category of decision variables includes infinitely many parameters in the cost model; i.e., h_j for $j = 1, 2, \ldots$. The constant IHI/SPI proposed by RB ($\int_{t_{j-1}}^{t_j} r(t)dt = \int_0^{t_1} r(t)dt$ which is equivalent to $\bar{F}(t_j) = (\bar{F}(h_1))^j$ and $p_j = p_1$) induces that the jth length of sampling intervals is computed through a function of the first time interval h_1 and j, i.e., obtaining the optimal design parameters n, L and h_j will be reduced to obtaining n, L and h_1. Then, the sampling intervals h_j for $j = 2, 3, \ldots$ can be obtained from h_1. This restriction indicates that the length of sampling intervals increases as the failure rate function decreases, stabilizes for time periods when the failure is constant, and reduces as the failure rate function increases (Fani et al. 2019).

For deteriorating processes, the RB's constant IHI/SPI restriction induces decreasing sampling intervals when the age of the system rises, or results in more frequent samplings with increasing in the failure rate of the shock model, and finally achieves lower value for ECT in comparison to USS-based cost model. As an example, for processes with IFR Weibull PFM with scale parameter $\lambda > 0$ and shape parameter $\nu \ge 1$, $f(t) = \lambda\nu t^{\nu-1}\exp(-\lambda t^{\nu})$, $F(t_j) = 1 - \lambda {t_j}^{\nu}$, $E(T_a) = \left(\frac{1}{\lambda}\right)^{\frac{1}{\nu}}\Gamma(1+\frac{1}{\nu})$, and $p_j = 1 - \exp(-\lambda({t_j}^{\nu} - {t_{j-1}}^{\nu}))$, the condition $p_j = p_1$ yields $t_j = \mathrm{j}^{\frac{1}{\nu}} h_1$ and $h_j = (\mathrm{j}^{\frac{1}{\nu}} - (j-1)^{\frac{1}{\nu}})h_1$ where h_j is a decreasing sequence; $h_1 > h_2 > h_3 > \cdots$. This approach can be adapted for any IFR lifetime distribution for finding the decreasing inspection scheme by which the IHI/SPI will be maintained constant in each sampling interval.

Although, the NSS with constant IHI/SPI achieves to a lower ECT in comarison with USS, it is reasonable to ask whether this approach is the best sampling scheme by which the cost model is minimized. In other words, the question is how to find a/an sampling/inspection scheme from the space of all the functional restrictions on the sampling intervals by which the ECT, obtained from (1) and (2), is minimized. The intuitive companionship between the sampling frequencies and the failure rate of a process can be discussed by investigating various functional constraints on choosing the length of sampling intervals and their effects on the optimal design of control charts.

In this paper, for more illustration of the problem, we induce 4 different decreasing procedures of sampling intervals on the optimization problem assuming normal QCD and IFR Weibull PFM. In other words, applying for deteriorating processes, we impose 4 different non-uniform functional constraints directly on the sampling intervals by which the length of intervals decreases over time. The purpose is to compare the procedures regarding to their corresponding value of the optimal ECT. In addition they will be compared with the NSS proposed by RB and the USS. The proposed procedures are as folow:
A: $h_j = a^{j-1} h_1;\ \ 0 < a \le 1,$
B: $h_j = \frac{1}{j^a} h_1;\ \ 0 \le a,$
C: $h_j = \frac{1}{(1+\frac{a}{j})^{j-1}} h_1;\ \ 0 \le a,$
D: $h_j = \frac{1}{1+a ln(j)} h_1;\ \ 0 \le a,$
where a is a scalar whose manipulation in each procedure results in various sequences of the that structure. By considering $a = 1$, the procedure A is equivalent to USS. The same holds for procedures B, C, and D by $a = 0$. The obtained sequences of IHI and SPI based on the above procedures are not necessarily decreasing or even constant as the system gets old. The jth sampling interval will be corresponded by a pair (h_j, p_j) for $j = 1, 2, ...$ ((h, p) in the case of USS with exponential PFM, (h, p_j) in the case of USS, and (h_j, p) in the case of NSS with constant IHI/SPI).

3. Numerical Studies

In this section, we demonstrate comparison of different procedures of sampling intervals, including the USS, RB's NSS, and 4 decreasing inspection restrictions, regarding to the optimal valuse of the ECT. Moreover, we present their corresponding results of statistical criteria, including the false alarm rate α, the power of detecting the occurrence of the asignable cause $1 - \beta$, the average number of sampling when the process is in contrl S, and th average number of sampling when the process is out-of-control untill the true alarm ARL_2.

3.1 $\bar{X}$ Control Chart

By far, the most attractive chart to monitor process mean across time is $\bar{X}$ control chart. The control limits of an $\bar{X}$ control chart are considered at $\mp L$ standard deviations limit from the average; i.e., $\mu \mp L\frac{\sigma}{\sqrt{n}}$, where the standard deviation σ is assumed fix through the time. Samples of size n are taken from the process output at time intervals of h_j $(j = 1, 2, ...)$ hours and then the value of sample mean will be plotted over the chart. Observing sample averages outside the control limits of the chart is a signal that the process is out of control. Detecting the assignable cause means the signal was true, otherwise it is considered as a false alarm.

The sampling subgroup measurements and subsequently the statistic sample mean are conventionally assumed to be normally distributed when one design a control chart to monitor a process. Under this assumption, the type I error probability α and the type II error probability β are calculated as follows:

$$\alpha = Pr\left(\bar{X} > \mu_0 + L\frac{\sigma}{\sqrt{n}}\middle|\mu = \mu_0\right) + Pr\left(\bar{X} < \mu_0 - L\frac{\sigma}{\sqrt{n}}\middle|\mu = \mu_0\right) = 2\Phi(-L), \qquad (3)$$

and

$$\beta = Pr\left(\mu_0 - L\frac{\sigma}{\sqrt{n}} \le \bar{X} \le \mu_0 + L\frac{\sigma}{\sqrt{n}}\middle|\mu = \mu_0 + \delta\sigma\right) = \Phi\left(L - \delta\sqrt{n}\right) - \Phi\left(-L - \delta\sqrt{n}\right), \qquad (4)$$

where Φ is the standard normal cumulative distribution function. The equations (3) and (4) do not depend on the mean and the standard deviation of the quality data when we assume that the original data is normally distributed.

For computational purposes of implementing the economic design of $\bar{X}$ control chart, we consider the input values of the cost and time parameters of the process from Banerjee and Rahim (1988); i.e., $Z_0 = 0.25$, $Z_1 = 1$, $a = \$20$, $b = \$4.22$, $D_0 = \$50$, $D_1 = \$950$, $Y = \$500$, and $W = \$1100$. In addition, we consider $\delta = 0.5$. The only values considered in this paper for the Weibull PFM parameters are two cases of $\lambda = 0.05$, $\nu = 2$, and $\lambda = 0.01$, $\nu = 2$. The following tables and figures clarify the comparison of different sampling procedures regarding to both economic and statistical criteria of the economic design of $\bar{X}$ control chart. The yellow color in tables and figures indicates the USS model, the green one shows the best state of the proposed sampling scheme (SS), the grey one is for the RB's NSS, and the red color indicates all the cases that the procedure is worst than the classical sampling method; i.e., the USS.

Table 2. Economic design parameters based on inspection procedure A when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 3.96}$

SS	n	h 1	P 1	L	**ECT**	ET	S+ARL2	alpha	1-beta
RB	19	2.90	0.344	1.36	442.71	5.94	1.91+1.26	0.174	0.794
Uniform (a=1)	23	1.50	0.107	1.34	467.17	6.07	2.13+1.17	0.181	0.856
a=0.95	22	1.56	0.114	1.36	461.30	6.00	2.21+1.19	0.174	0.838
a=0.9	21	1.66	0.129	1.37	456.09	5.96	2.22+1.22	0.170	0.821
a=0.87	23	2.03	0.186	1.29	455.81	6.04	1.79+1.16	0.196	0.866
a=0.85	23	2.29	0.230	1.23	457.41	6.13	1.51+1.14	0.217	0.878
a=0.8	25	2.97	0.356	1.08	465.75	6.33	1.07+1.08	0.279	0.922
a=0.78	24	3.06	0.374	1.29	466.84	6.12	1.76+1.14	0.197	0.877
a=0.7	18	3.74	0.503	1.27	478.31	6.68	1.57+1.25	0.204	0.803
a=0.6	16	4.35	0.612	1.32	496.25	6.93	1.42+1.33	0.187	0.752

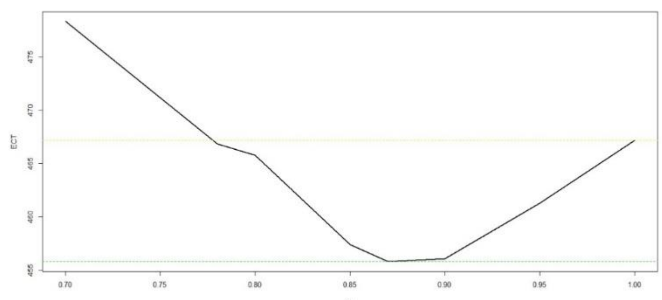

Figure 2. Economic design ECT values based on inspection procedure A when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 3.96}$

Table 2 and Figure 2 show that the procedure A, in its best state of $a = 0.87$ with ECT=$455.81 per hour, could not attain the RB's NSS with constant IHI/SPI whose ECT value equals $442.71. For $a = 1$ and $a = 0.78$, procedure A did as the same as the USS. For $0.78 < a < 1$, procedure A is better than the USS, however, as $a < 0.78$ decreases, the ECT increases and the USS is superior to procedure A. Procedure B seems to act better than procedure A, as it is clear by Table 3 and Figure 3, but it could not also attain the RB. Its best state is achieved by $a = 0.75$ with ECT=$444.48 per hour. For $0 < a < 1.32$, procedure B is better than the USS, however as $a > 1.32$ increases, the ECT also increases, and so the USS dominates procedure B. In a nutshell, both procedures A and B are not the preferable procedures for a quality practitioner, as the RB's approach yield to a lower ECT.

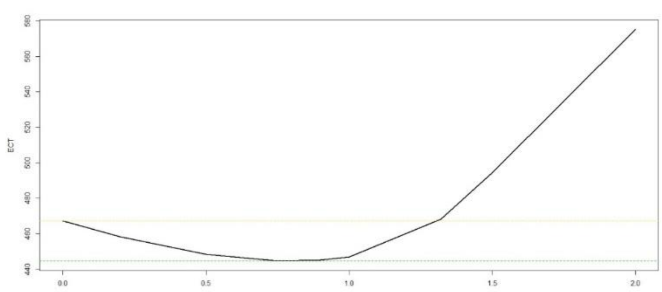

Figure 3. Economic design ECT values based on inspection procedure B when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 3.96}$

Table 3. Economic design parameters based on inspection procedure B when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 3.96}$

SS	n	h 1	P 1	L	**ECT**	ET	S+ARL2	alpha	1-beta
RB	19	2.90	0.344	1.36	442.71	5.94	1.91+1.26	0.174	0.794
Uniform (a=0)	23	1.50	0.107	1.34	467.17	6.07	2.13+1.17	0.181	0.856
a=0.2	22	1.74	0.141	1.34	458.27	6.00	2.07+1.19	0.180	0.842
a=0.5	21	2.13	0.203	1.35	448.28	5.94	1.98+1.21	0.178	0.827
a=0.6	21	2.28	0.229	1.35	446.18	5.92	1.95+1.21	0.177	0.827
a=0.75	20	2.52	0.272	1.34	444.48	5.94	1.89+1.23	0.179	0.814
a=0.9	20	2.80	0.325	1.32	445.01	5.97	1.80+1.22	0.183	0.817
a=1	20	3.03	0.367	1.32	446.95	6.02	1.71+1.22	0.188	0.821
a=1.32	21	4.10	0.569	1.18	468.21	6.41	1.20+1.15	0.240	0.868
a=1.5	21	4.91	0.700	1.07	494.56	6.85	1.98+1.12	0.284	0.889
a=2	20	6.84	0.904	1.09	575.22	8.33	1.10+1.14	0.274	0.874

In the case of procedure C, when $a = 2$, the ECT is approximately equal to that of RB if the PFM parameters are $\lambda = 0.05$ and $\nu = 2$. As we can see from Table 4 and Figure 4, for the values $a > 4.1$, the procedure C is dominated by the USS. Considering a process with $\lambda = 0.01$ and $\nu = 2$, procedure C attains a value 311.04 for ECT, lower than that of RB which is 311.86, when $a = 1.8$. Based on Table 5 and Figure 5, one can conclude procedure C dominates RB, for values of $1.45 < a < 2.15$, otherwise RB is superior. The USS, in this case, is also more economical than procedure C when $a > 3.45$.

Table 4. Economic design parameters based on inspection procedure C when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 3.96}$

SS	n	h 1	P 1	L	**ECT**	ET	S+ARL2	alpha	1-beta
RB	19	2.90	0.344	1.36	442.71	5.94	1.91+1.26	0.174	0.794
Uniform (a=0)	23	1.50	0.107	1.34	467.17	6.07	2.13+1.17	0.181	0.856
a=0.05	23	1.55	0.113	1.33	466.01	6.06	2.11+1.17	0.182	0.856
a=0.5	22	1.90	0.165	1.32	456.58	6.02	1.95+1.18	0.186	0.847
a=0.9	24	3.67	0.126	1.49	312.97	11.18	3.54+1.20	0.137	0.832
a=1	22	2.21	0.217	1.33	448.92	5.95	1.89+1.18	0.185	0.846
a=2	**20**	**2.72**	**0.309**	**1.35**	**442.94**	**5.92**	**1.90+1.23**	**0.176**	**0.812**
a=3	18	3.25	0.410	1.36	448.56	6.03	1.89+1.29	0.172	0.776
a=4.1	18	3.96	0.543	1.34	467.14	6.30	1.76+1.28	0.180	0.783
a=5	18	4.66	0.662	1.29	491.31	6.68	1.57+1.25	0.197	0.797
a=6	4	10.74	0.685	1.95	455.25	13.72	16.42+5.77	0.052	0.173

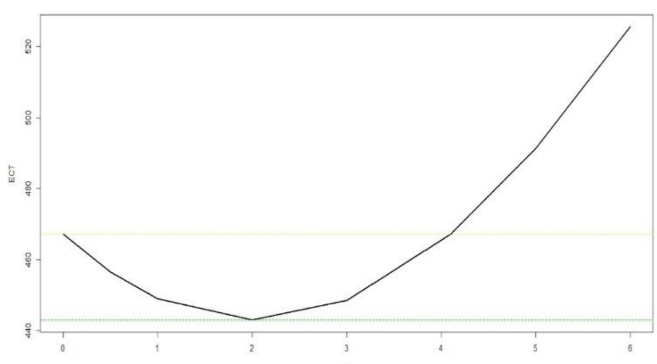

Figure 4. Economic design ECT values based on inspection procedure C when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 3.96}$

Table 5. Economic design parameters based on inspection procedure C when $\boldsymbol{\lambda = 0.01}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 8.86}$

SS	n	h 1	P 1	L	ECT	ET	S+ARL2	alpha	1-beta
RB	22	4.95	0.217	1.51	311.86	11.21	3.60+1.25	0.131	0.799
Uniform (a=0)	25	1.97	0.038	1.49	330.29	11.35	4.00+1.18	0.137	0.844
a=0.05	25	2.04	0.041	1.49	329.44	11.34	3.96+1.18	0.137	0.845
a=0.5	25	2.69	0.070	1.48	322.03	11.28	3.67+1.18	0.140	0.847
a=1	25	3.37	0.107	1.48	315.49	11.20	3.52+1.18	0.139	0.846
a=1.45	24	3.97	0.146	1.49	311.74	11.17	3.51+1.20	0.136	0.831
a=1.8	**23**	**4.32**	**0.170**	**1.50**	**311.04**	**11.18**	**3.55+1.23**	**0.134**	**0.816**
a=2.15	23	4.72	0.200	1.51	311.79	11.18	3.62+1.23	0.131	0.813
a=3	21	5.53	0.264	1.54	315.19	11.23	3.97+1.29	0.123	0.773
a=5	17	6.35	0.332	1.61	327.31	11.41	4.86+1.48	0.108	0.675
a=3.45	21	6.37	0.333	1.53	330.06	11.52	3.85+1.29	0.127	0.777
a=4	21	7.2	0.405	1.52	345.37	11.81	3.84+1.28	0.128	0.779
a=5	22	8.96	0.552	1.49	383.98	12.61	3.68+1.24	0.136	0.804

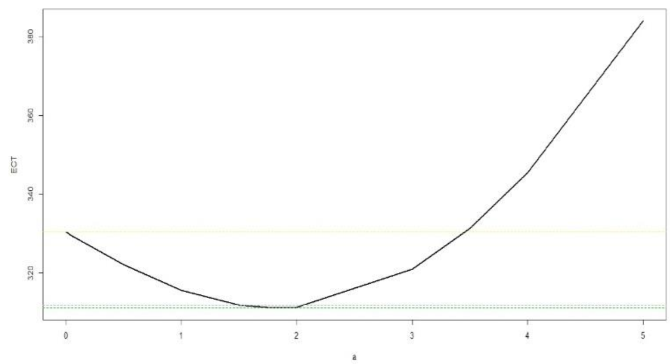

Figure 5. Economic design ECT values based on inspection procedure C when $\boldsymbol{\lambda = 0.01}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 8.86}$

Similar results can be obtained by discussing Tables 6 and 7 for the case of procedure D. In both cases of $\lambda = 0.05$ and $\lambda = 0.01$, procedure D attains an ECT value lower than that of RB. The coefficients of a for which this procedure got an ECT's quantity equal to those of RB and the USS are specified in Figures 6 and 7.

Table 6. Economic design parameters based on inspection procedure D when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 3.96}$

SS	n	h 1	P 1	L	ECT	ET	S+ARL2	alpha	1-beta
RB	19	2.90	0.344	1.36	442.71	5.94	1.91+1.26	0.174	0.794
Uniform (a=0)	23	1.51	0.107	1.34	467.17	6.07	2.13+1.17	0.181	0.856
a=0.05	23	1.56	0.115	1.34	464.92	6.04	2.11+1.17	0.181	0.856
a=0.5	22	2.02	0.185	1.33	451.88	5.96	1.96+1.18	0.182	0.844
a=0.9	21	2.34	0.239	1.34	446.22	5.94	1.89+1.20	0.182	0.831
a=1	21	2.40	0.251	1.34	445.33	5.93	1.88+1.20	0.181	0.830
a=1.5	20	2.68	0.301	1.35	442.83	5.92	1.88+1.23	0.178	0.813
a=1.8	**20**	**2.79**	**0.322**	**1.35**	**442.5**	**5.92**	**1.90+1.23**	**0.176**	**0.811**
a=2.2	19	2.96	0.354	1.37	442.91	5.93	1.94+1.26	0.171	0.792
a=3	18	3.20	0.400	1.40	445.56	5.96	2.05+1.31	0.163	0.766
a=5	15	3.61	0.479	1.46	456.01	6.09	2.37+1.46	0.145	0.685
a=7.05	13	3.90	0.532	1.50	467.02	6.21	2.69+1.62	0.132	0.618
a=10	11	4.22	0.589	1.56	481.59	6.38	3.12+1.85	0.119	0.541
a=50	4	6.00	0.835	1.72	561.95	7.63	5.09+4.16	0.086	0.240

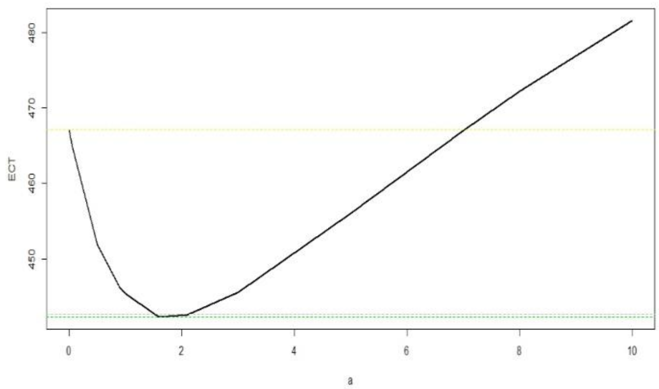

Figure 6. Economic design ECT values based on inspection procedure D when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 3.96}$

Table 7. Economic design parameters based on inspection procedure D when $\boldsymbol{\lambda = 0.01}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 8.86}$

SS	n	h 1	P 1	L	**ECT**	ET	S+ARL2	alpha	1-beta
RB	22	4.95	0.217	1.51	311.86	11.21	3.60+1.25	0.131	0.799
Uniform (a=0)	25	1.97	0.038	1.49	330.29	11.35	4.00+1.18	0.137	0.844
a=0.05	25	2.08	0.042	1.49	328.20	11.32	3.96+1.18	0.137	0.844
a=0.5	25	3.01	0.087	1.49	317.26	11.20	3.66+1.18	0.137	0.845
a=0.9	24	3.67	0.126	1.49	312.97	11.18	3.54+1.20	0.137	0.832
a=1	24	3.81	0.135	1.49	312.33	11.18	3.53+1.20	0.137	0.832
a=1.1	24	3.91	0.142	1.49	311.82	11.17	3.52+1.20	0.136	0.831
a=1.5	**23**	**4.40**	**0.176**	**1.49**	**310.82**	**11.18**	**3.54+1.23**	**0.134**	**0.816**
a=2.2	22	5.02	0.223	1.52	311.78	11.19	3.69+1.26	0.129	0.796
a=3	21	5.53	0.264	1.54	315.19	11.23	3.97+1.29	0.123	0.773
a=5	17	6.35	0.332	1.61	327.31	11.41	4.86+1.48	0.108	0.675
a=5.45	17	6.49	0.344	1.62	330.21	11.43	5.08+1.49	0.105	0.670
a=10	12	7.45	0.426	1.72	356.63	11.82	7.27+1.99	0.085	0.503
a=50	4	10.74	0.685	1.95	455.25	13.72	16.42+5.77	0.052	0.173

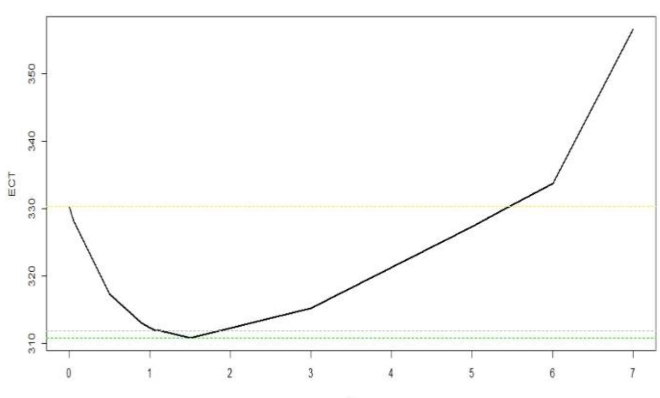

Figure 7. Economic design ECT values based on inspection procedure D when $\boldsymbol{\lambda = 0.01}$, $\boldsymbol{\nu = 2}$, and $\boldsymbol{ET_a = 8.86}$

3.2 T^2 Control Chart

Many products may have more than one quality characteristic to be monitored before submitting to the customer. For an example, Rahim and Raouf (1988) considered a process that produces a type of cylinder with an inner diameter and an outer diameter. These two dimensions determine the effectiveness of the cylinder together. In these cases, control of the process mean vector can be performed by using the widely known Hotelling T^2 control chart. The multivariate quality characteristic $\boldsymbol{X}$ has a multivariate normal distribution, denoted by $N_k(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu}$ is the $k \times 1$ mean vector and Σ is the $k \times k$ known covariance matrix of $\boldsymbol{X}$. Let the process mean be $\boldsymbol{\mu} = \boldsymbol{\mu_0} = [\mu_{01}, \ldots, \mu_{0k}]^T$, when the process is in-control, and shifts from $\boldsymbol{\mu_0}$ to $\boldsymbol{\mu_1} = \boldsymbol{\mu_0} + \boldsymbol{\delta}$ after the occurrence of the assignable cause, where $\boldsymbol{\delta} \neq \boldsymbol{0}$, and $\boldsymbol{\delta} = [\delta_1, \ldots, \delta_k]^T$. Random samples of size n, denoted by $\boldsymbol{X_1}, \boldsymbol{X_2}, \ldots, \boldsymbol{X_n}$ are taken from the output of the process in consequent time intervals h_j $(j = 1, 2, \ldots)$ hours. The statistic $T^2 = n(\boldsymbol{\bar{X}} - \boldsymbol{\mu_0})^T\Sigma^{-1}(\boldsymbol{\bar{X}} - \boldsymbol{\mu_0})$ that has chi square distribution with k degrees of freedom (denoted by χ_k^2) in the in-control state will be computed. Once $\chi_k^2 > \chi_{\alpha,k}^2$, it is indicated that the process is out of control statistically with the Type I error probability of α, where $L = \chi_{\alpha,k}^2$ is the upper percentage point of chi square distribution with k degrees of freedom. L is considered as the upper control limit of T^2 control chart, whereas the T^2 statistic is non-negative, the lower control limit is zero. In the out-of-control state, T^2 has a non-central chi square distribution with k degrees of freedom and non-central parameter $\xi = n\boldsymbol{\delta}^T\Sigma^{-1}\boldsymbol{\delta} = n(\boldsymbol{\mu} - \boldsymbol{\mu_0})^{\boldsymbol{T}}\Sigma^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu_0})$ (donated by $\chi_{\xi,k}^2$). Henceforth, one can conclude $\alpha = \Pr(\chi_k^2 > L)$ and $\beta = \Pr(\chi_{\xi,k}^2 < L)$. Montgomery and Klatt (1972) were the first to address the design of T^2 charts from an economic point of view. A review of the multivariate control chart is given by Lowry and Montgomery (1995).

For numerical illustration in the case of multivariate QCD, we just consider the procedure D to present the result of ED. The process time and cost parameters, beside the input parameters of mean and shift vectors are based on the numerical study of Yang and Rahim (2005). Similar to the case of univariate QCD, procedure D dominates the NSS with constant IHI/SPI. For the values of $a = 0$ and $a = 5$, procedure D is the same as the USS, and for $a > 5$, it even is worst than the USS. For $0 < a < 5$, the proposed procedure attains an ECT lower than that of the USS, and for values of $1 < a < 2$, it is superior to RB. The results are presented in Table 8 and Figure 8. For the procedures A, B, and C, the results in the case of T^2 control chart are similar to those of $\bar{X}$ control chart shown in Tables 2, 3, 4, and 5.

Table 8. Economic design parameters of $\boldsymbol{T^2}$ control chart based on inspection procedure D when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$

SS	n	h 1	P 1	L	**ECT**	ET	S+ARL2	alpha	1-beta
RB	12	2.45	0.259	6.27	394.49	5.58	2.86+1.15	0.043	0.871
Uniform (a=0)	13	1.06	0.055	6.26	409.33	5.65	3.24+1.11	0.044	0.899
a=0.5	13	1.55	0.114	6.23	398.52	5.58	2.96+1.11	0.044	0.899
a=1	13	1.92	0.169	6.23	394.53	5.56	2.85+1.11	0.044	0.899
a=1.5	**12**	**2.20**	**0.215**	**6.25**	**393.52**	**5.58**	**2.84+1.15**	**0.044**	**0.872**
a=2	12	2.42	0.254	6.28	394.17	5.58	2.89+1.15	0.043	0.871
a=4	11	2.98	0.359	6.44	404.06	5.65	3.30+1.20	0.040	0.830
a=5	10	3.17	0.396	6.51	410.33	5.71	3.53+1.27	0.038	0.785
a=6	10	3.34	0.427	6.57	416.60	5.75	3.77+1.28	0.037	0.782
a=7	9	3.48	0.453	6.63	481.59	5.82	3.99+1.37	0.036	0.729

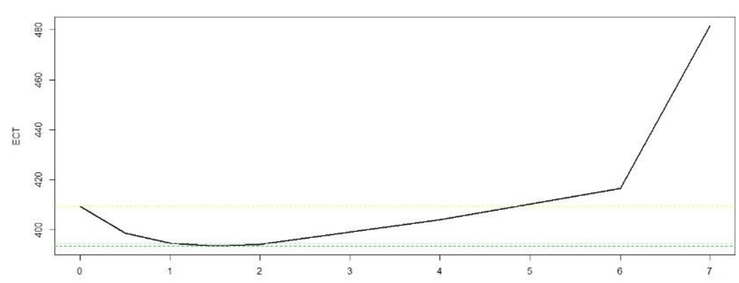

Figure 8. Economic design ECT values of $\boldsymbol{T^2}$ control chart based on inspection procedure D when $\boldsymbol{\lambda = 0.05}$, $\boldsymbol{\nu = 2}$

4. Conclusion

As a process deteriorates over time, the reasonable approach for sampling over time is to decrease the time intervals. Various decreasing inspection scheme are applied in this paper for IFR processes with Wiebull distribution and the optimal design outputs of their corresponding economic design are compared to the non-uniform sampling scheme with constant integrated hazard and to the uniform sampling scheme. All the proposed procedures, like the constant integrated hazard, simplify the optimization problem by inducing a function in which the jth sampling interval is computed through a function of the first interval h_1 and j. In some cases, the proposed procedures are slightly better than the constant integrated hazard approach. Further investigation on the mathematical modelling is needed to find the best non-uniform sampling scheme in the space of all sampling strategies such that the cost model is minimized.

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