Contact Mechanics for a Ball Dropped on an Infinite Plate, an Update

Carlos Garcia-Mendoza and Neyfe Sablón
Instituto de Posgrado
Universitdad Técnica de Manabí (UTM)
Portoviejo, Ecuador
daniel.garcia@utm.edu.ec, neyfe.sablon@utm.edu.ec

Abstract

Contact mechanics has been largely used for the prediction of the behavior of material internal stresses due to external interactions. The analysis of these problems in the elastic regime was first done by Hertz, who focused on the behavior of the stresses in the near vicinities of the contact. The case of a ball interacting with an infinite plate is a special case of Hertz's theory, which was originally designed for the interaction between two spheres. For this case, classical equations for the prediction of maximum normal pressure and maximum reaction force are still being used, which date as back as 1948. These equations however, seem to significantly overestimate the maximum normal pressure generated by the interactions of these two bodies, an observation also induced by the authors which first derived these results. Besides the presence of questionable theoretical basis, a significant proportion of the literature still takes these models as the only and irrefutable source of theoretical estimations. This paper shows the derivation of new contact mechanics equations for the case of a ball being dropped on an infinite plate of the same material. The derivation of these equations takes Hertz's theory as a fundamental preliminary. It assumes a sphere of radius R falling from a vertical distance h from the plate. The contact is assumed to be elastic, and so the Work-Energy Theorem was used to calculate the work to be done by the plate on the sphere to decelerate it. Given that the integral solution for the work experienced by the sphere using this method is significantly complex, a series expansion around the origin is used. Additionally, given that the change in the independent variable is significantly small, the first term of this expansion results in a good approximation of the original solution. Therefore, the proposed model is:

$$p_{max}(v) = \frac{E\left\{R^2 - \left[R - \left(\frac{5\rho\pi R^{2.5}v^2}{3\sqrt{2}E}\right)^{0.4}\right]^2\right\}^{0.5}}{R(1-\sigma^2)\pi}$$

Where $p_{max}(v)$ is the maximum normal stress experienced, E is the Young's modulus, σ is the Poisson's ratio, v the speed of the ball right before contacting the plate and ρ the material's density. This model, compared to the classical equations, takes into account the ball radius and predicts much lower maximum normal pressures by predicting a lager contact area, as expected.

Keywords
Contact Mechanics, Dynamic Contact, Hertz's Theory, Stress Distribution.

1. Introduction
In order to understand how materials fail under specific load cases, it is important to investigate how the internal stresses behave and what are the loads that induce them. When the internal stresses reach the critical material resistance, several mechanisms of failure could occur, depending on the material properties and load scenario. The goal is to know when, where and how these stress levels are reached, giving us the opportunity of preventing material failure. For this reason this topic has been extensively studied in the literature, with the aim of understanding the flow of stresses within a body under different loads.

Gahr (1987), who compiled some of the most relevant results regarding material mechanics in his work, shows several important equations for the prediction of material behavior. In particular, he shows the equations that predict the behavior of internal stresses generated by static loads between bodies of the same material (between spheres, between a sphere and a plate, between cylinders). It is also shown the expression to calculate the magnitudes of the reaction forces generated under dynamic interactions. These equations also appear in a work from Tabor (1951), who in turn cites a paper from Davies (1949) called *The Determination of Static and Dynamic Yield Stresses Using a Steel Ball.* This work, in most of the literature regarding the prediction of internal stresses due to contacting bodies, is taken as general reference, making deliberate use of the equations shown in it. Davies makes use of Hertz's theory as a starting point for the construction of the equations that predict internal stresses and reaction forces. In his work, the equations for the prediction of the internal stresses within an infinite steel plate generated by pressing a steel ball over it are shown (static scenario). Moreover, Davies tries to investigate how the internal stresses behave in a dynamic case, that is, the stresses generated withing a steel plate due to a ball striking its surface after free fall. An important point to raise is that, after a detailed review of these results, it is possible to see that the equations generated for the dynamic case seem to overestimate real scenarios. For example, using these equations, the stress generated by a steel ball of radius 1.27cm falling from a height of just 15cm (dynamic case) would generate the same pressure on the surface of a plate as a ball of the same radius pressed against a plate with a load of 527kg (static case). This aspect is also recognized by Davies himself, who states: 'A striking feature brought out by the data given in this table is the surprisingly large values of the normal pressure given by small values of velocity and height of drop'.

In this work, we explore the derivation of new equations for the prediction of the internal stresses withing an infinite plate, generated by the impact of a ball after free fall. For this, Hertz's theory is used as a fundamental basis, comparing the results obtained by Davies and the new contact model proposed. The first section of this work shows the preliminary theory, to finally show the derivation of the new contact mechanics equations for the dynamic case specified above.

1.1 Objectives

The objectives of this work are:

- To derive new equations for the prediction of maximum pressures and forces experienced by an infinite plate hit by a sphere of the same material after free fall.
- To compare the new proposed model with the one proposed by Davies.

2. Background

Hertz (1882) published the first extensive work regarding contact mechanics between solid bodies. He showed the relationship between material properties, their geometry and loads applied on the contacting bodies. These relationships allowed to estimate important parameters such as contact pressure, contact force, contact projected area, etc. Medeiros (2002) claims that one of the strongest contributions of Hertz theory was to demonstrate that it is only necessary to know the geometry of the material and the elastic properties it has to estimate the contact area and the magnitudes of the force and pressure present when two bodies are pressed together. It is important to note that Hertz theory, as stated by de Souza and Galvao (2011), makes important assumptions such as:

1. The materials are homogeneous.
2. The Yield stress of the material is not exceeded
3. The contact stress is only generated by the load applied to it, and no tangential forces (due to friction or other means) are present.
4. The contact area is significantly small compared to the dimensions of the bodies involved.
5. The contacting bodies are at equilibrium conditions.
6. Surface roughness (and therefore friction) is neglected.

Or as Ingraffea and Wawrzynek (2007) state, Hertz theory makes the assumption that material surfaces are continuous, nonconforming, strains are small and that each body can be considered as a half-space. This theory therefore perfectly suits the considered scenario, where a ball strikes a flat surface after free fall, considering that very small strains are experienced, small contact areas are expected (as metallic materials are used), materials are homogeneous, etc. It is important to note that these assumptions are made given that, as Hertz (1882) claims: 'We can confine our attention to that part of each body which is very close to the point of contact, since here the stresses are extremely great compared with those occurring elsewhere'. Once this is assumed, as Johnson (1982) pointed out, it is possible to find the deformations and stresses by neglecting the curvature of the surfaces in contact, and treating each as half-spaces. Some of the most relevant results from Hertz theory, as described by Wang and Chung (2013) will be shown below. Consider a normal loading area Ω, a material with Young's modulus E and Pisson's ratio v, integration of the deformation point (x, y) caused by a normal stress $p(\xi, \eta)$ over Ω results in the total deformation modeled by the following expression.

$$u_z(x, y) = \frac{(1 - v^2)}{\pi E} \iint_\Omega \frac{p(\xi, \eta)}{\sqrt{(x - \xi)^2 + (x - \eta)^2}} d\xi d\eta \qquad \text{Equation 1}$$

This expression was derived by Boussinesq (1885), and if contact is not assumed to be frictionless, the relative displacement u_z can be formulated in terms of the frictional shear stress q_x, q_y and pressure p as:

$$u_z(x, y) = \iint_\Omega \left\{\frac{x - \xi}{\pi\rho^2\mu'} \cdot q_x(\xi, \eta) + \frac{y - \eta}{\pi\rho^2 u'} \cdot q_y(\xi, \eta) + \frac{1}{\pi\rho E^*} \cdot p(\xi, \eta)\right\} d\xi d\eta \qquad \text{Equation 2}$$

Where $\rho = \sqrt{(x - \xi)^2 + (x - \eta)^2}$, $\frac{1}{E'} = \frac{(1+v_1)}{E_1} + \frac{(1+v_2)}{E_2}$, $\frac{1}{E^*} = \frac{(1-v_1^2)}{E_1} + \frac{(1-v_2^2)}{E_2}$, $\frac{1}{\mu'} = \frac{(1+v_1)(1-v_2)}{2E_1} - \frac{(1-v_1)(1+v_2)}{2E_2}$ and $\frac{1}{\mu^*} = \frac{v_1(1+v_1)}{E_1} + \frac{v_2(1+v_2)}{E_2}$ where the subscripts refer to the two bodies involved in the contact. If a ball of equivalent radius R_e is deformed by a flat surface with a load W, the contact area should be circular. In this case, the summation of the elastic deformation and the original separation can be expressed as:

$$\delta = u_z(r) + z(r) \qquad \text{Equation 3}$$

Where:

$$z(r) \approx \frac{r^2}{2R_e} \qquad \text{Equation 4}$$

One of the parabolic pressure distributions that fit the model, considering that a maximum pressure (called Hertzian pressure P_h) is obtained at the center of the contact circle of radius a, is the following:

$$p(x, y) = p(r) = P_h \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}} \qquad \text{Equation 5}$$

Therefore, the maximum pressure can be obtained by integrating Equation 5 over a circular area and finally obtaining the Hertz equations for the calculation of pressure distribution:

$$p(x, y) = \frac{3W}{2\pi a^2} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}} \qquad \text{Equation 6}$$

Maximum pressure:

$$P_h = \frac{1}{\pi} \sqrt[3]{\frac{6WE^{*2}}{R_e^2}} \qquad \text{Equation 7}$$

And contact radius:

$$a = \sqrt[3]{\frac{2WR_e}{4E^*}} \qquad \text{Equation 8}$$

Classical Hertz theory has then used to estimate the behavior of pressure and reaction forces due to contacting bodies. For the derivation of the equations that predict such behavior, as Whang (2011) states, it is assumed that: (a) the stress field generated by the elastic impact is identical to that generated in static conditions and (b) the reacting normal force is due to the deceleration of the bodies impacting each other. This method has been largely used for the study of the reaction of the material under shot peening for example, where parabolic pressure distributions together with Hertz theory are also used (Fathallah et al. 1998).

5. Results and Discussion

For the derivation of the new dynamic case equations, Hertz's contact theory will be used. In particular, we take the equations derived for the static case of a sphere pressed against a plate as a starting point which assumes A ball of radius R pressed against a plate with a load P, as shown in Figure 1.

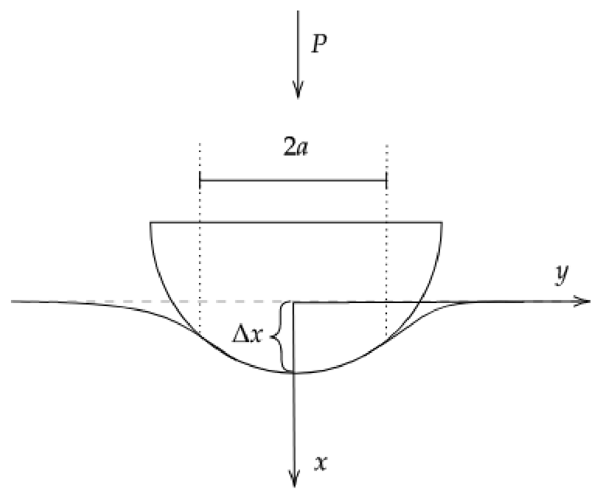

Figure 1. Indentation produced by a sphere on a plate when a load P is applied

The interaction between these two bodies generates an indentation on the plate, with a projected contact circle of radius a. Hertz's theory states that the radius of the projected circle can be calculated with the following equation:

$$a^3 = \frac{1.5(1-\sigma^2)PR}{E} \qquad \text{Equation 9}$$

Where E is the Young's modulus of the plate and σ is its Poisson's ratio. The distribution of the normal pressure over the circle of contact follows the following law.

$$p = p'\sqrt{\frac{a^2 - r^2}{a}} \qquad \text{Equation 10}$$

Where p is the normal pressure at distance r from the centre of the circle of contact and p' is the maximum normal pressure in the circle of contact. It is then possible to derive the equation of maximum normal pressure as follows.

$$p' = \frac{1.145}{\pi}\left(\frac{E}{1-\sigma^2}\right)^{\frac{2}{3}}\left(\frac{P}{R^2}\right)^{\frac{1}{3}} \qquad \text{Equation 11}$$

For the dynamic case, in particular, it is assumed that a sphere of radius R falls from a distance h, and immediately before contacting the plate, the sphere has a linear velocity v, normal to the plate. The impact between the sphere and the plate produces an indentation as in Figure 2. It will also be assumed that the contact between the two bodies is purely elastic, and that the only force responsible for the deceleration of the sphere is the normal force resulting from the elastic interaction between the bodies. It is then possible to use the following result:

Work-Energy Theorem: The work done by the resultant force F acting on a particle as it moves from point A to point B along its trajectory is equal to the change in kinetic energy of the particle during the given displacement.

Considering this theorem and the scenario being studied, it is possible to state that the kinetic energy of the sphere right before impacting the plate is equal to the work done by the reaction force that the plate generates on the sphere to reduce its speed to zero. It is important to state that energy is not always conserved during impact of two bodies, but for this initial approach, this will be assumed as true, and that the net force acts purely along the trajectory of the sphere. Then, work can be calculated by the following integral:

$$W = \int_{x_1}^{x_2} P\, dx \qquad \text{Equation 12}$$

Where P is the force experienced by the object in the vertical direction defined in Figure 1 and $\Delta x = x_2 - x_1$. For simplicity, it will be assumed that x_1 is at the origin, and so any displacement x in the vertical direction can be calculated by:

$$x \cong R - \sqrt{R^2 - a^2} \qquad \text{Equation 13}$$

Then using this result in Equation 9 we have:

$$P(x) = \frac{E[R^2 - (R-x)^2]^{3/2}}{R[1.5(1-\sigma^2)]} \qquad \text{Equation 14}$$

Therefore, the work done by this force would be:

$$W = \int_{x_1}^{x_2} \frac{E[R^2 - (R-x)^2]^{3/2}}{R[1.5(1-\sigma^2)]} dx \qquad \text{Equation 15}$$

Then, since it is assumed that the work done should be equal the initial kinetic energy right before impact, as previously stated, we have:

$$\frac{4}{3}\pi\rho R^3 v^2 = \int_{x_1}^{x_2} \frac{E[R^2 - (R-x)^2]^{3/2}}{R[1.5(1-\sigma^2)]} dx \qquad \text{Equation 16}$$

Where ρ represents the density of the sphere and plate. Conveniently, the previous integral can be calculated from $x = 0$ (right before impact) to $x = d$ where d is the maximum vertical distance travelled by the sphere, as shown below.

$$\frac{4}{3}\pi\rho R^3 v^2 = \int_{0}^{d} \frac{E[R^2 - (R-x)^2]^{3/2}}{R[1.5(1-\sigma^2)]} dx \qquad \text{Equation 17}$$

The integral in the previous expression has the following solution:

$$\int \frac{E[R^2 - (R-x)^2]^{\frac{3}{2}}}{R[1.5(1-\sigma^2)]} dx = \frac{1}{8}\sqrt{x(2R-x)}\left(\frac{6R^{3.5}\sin^{-1}\left(\frac{\sqrt{x}}{\sqrt{2R}}\right)}{\sqrt{x\left(2-\frac{x}{R}\right)}} - 3R^3 - R^2x + 6Rx^2 - 2x^3\right)$$

Given that the exact solution of the integral introduces significant complexity to the expression, it was necessary to use a series expansion around $x = 0$ of the solution and since the displacement x is considerably small (due to very small elastic strains), the first term of the expansion would be a good approximation of it. Then, for significantly small displacements, the integral can be approximated as follows:

$$\int \frac{E[R^2-(R-x)^2]^{\frac{3}{2}}}{R[1.5(1-\sigma^2)]}dx \approx \frac{4}{5}\sqrt{2}R^{1.5}x^{2.5} \qquad \text{Equation 19}$$

Therefore, the expression that relates the kinetic energy of the sphere before impact and the work necessary to stop it during the collision is:

$$\frac{4}{3}\pi\rho R^3 v^2 = \int_0^d \frac{E[R^2-(R-x)^2]^{3/2}}{R[1.5(1-\sigma^2)]}dx \approx \frac{4}{5}\sqrt{2}R^{1.5}d^{2.5} \qquad \text{Equation 20}$$

$$\frac{4}{3}\pi\rho R^3 v^2 = \frac{4}{5}\sqrt{2}R^{1.5}d^{2.5} \qquad \text{Equation 21}$$

It is then possible to solve for d and obtain the equation of the maximum force experienced by the ball during the impact as a function of the speed before impact:

$$d = \left(\frac{5\rho\pi R^{1.5}v^2}{3\sqrt{2}E}\right)^{0.4} \qquad \text{Equation 22}$$

$$P(v) = \frac{E\left\{R^2-\left[R-\left(\frac{5\rho\pi R^{1.5}v^2}{3\sqrt{2}E}\right)^{0.4}\right]^2\right\}^{1.5}}{1.5R(1-\sigma^2)} \qquad \text{Equation 23}$$

Then, the maximum normal stress experienced by the plate can be approximated by:

$$p_{max}(v) = \frac{E\left\{R^2-\left[R-\left(\frac{5\rho\pi R^{1.5}v^2}{3\sqrt{2}E}\right)^{0.4}\right]^2\right\}^{0.5}}{R(1-\sigma^2)\pi} \qquad \text{Equation 24}$$

It is important to highlight that this proposed model takes into account the radius of the sphere. Also, it predicts that the maximum pressure that could be experienced during collision is achieved when the size of the contact circle equals the size of the sphere. The proposed model can be graphed as shown in Figure 2.

Figure 2. Graphical representation of proposed model

Contrary to Davies' model, it is possible to observe that the pressure is fairly dependent of the radius of the sphere. More specifically, Davies's model predicts that the maximum pressure experienced by the plate when a sphere of the same material hits is surface can be calculated with the following expression.

$$p' = \frac{1}{\pi}(2.5\pi\rho)^{\frac{1}{5}}\left(\frac{E}{1-\sigma^2}\right)^{4/5} v^{2/5} \qquad \text{Equation 25}$$

To illustrate the difference between these two models, Figure 3 can be used. Note that this figure takes into account the change in maximum pressure as a function of impact speed only, considering the specific case of a ball of radius $R = 1cm$ being dropped on a plate at different heights.

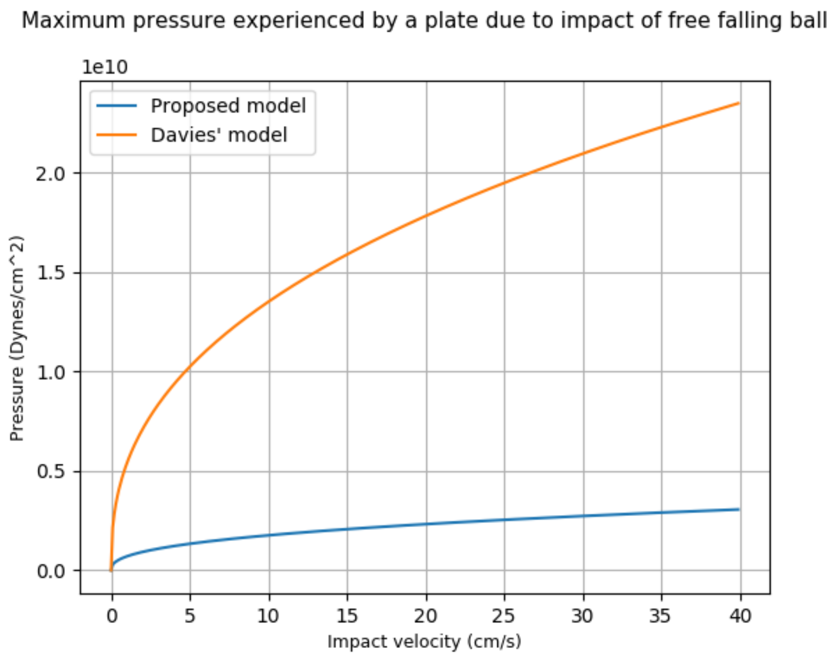

Figure 3. Comparison of the prediction of maximum normal pressure from proposed model with Davies' model assuming a ball of radius 1cm

From Figure 3 it is possible to observe significant differences between the two models. For instance, when the ball impacts a plate at 25cm/s, the pressure predicted by Davies would be approximately $1.9 * 10^{10} dynes/cm^2$ while the proposed model predicts a pressure of $6.9 * 10^{9} dynes/cm^2$. The main difference between the two models comes from the prediction of the normal force experienced during collision. The model proposed in this work predicts a much lower maximum force for the same experimental conditions. This is illustrated in Figure 4, where it is possible to observe that for any given impact velocity, the maximum normal force predicted by the proposed model is significantly lower than the one predicted by Davies' model.

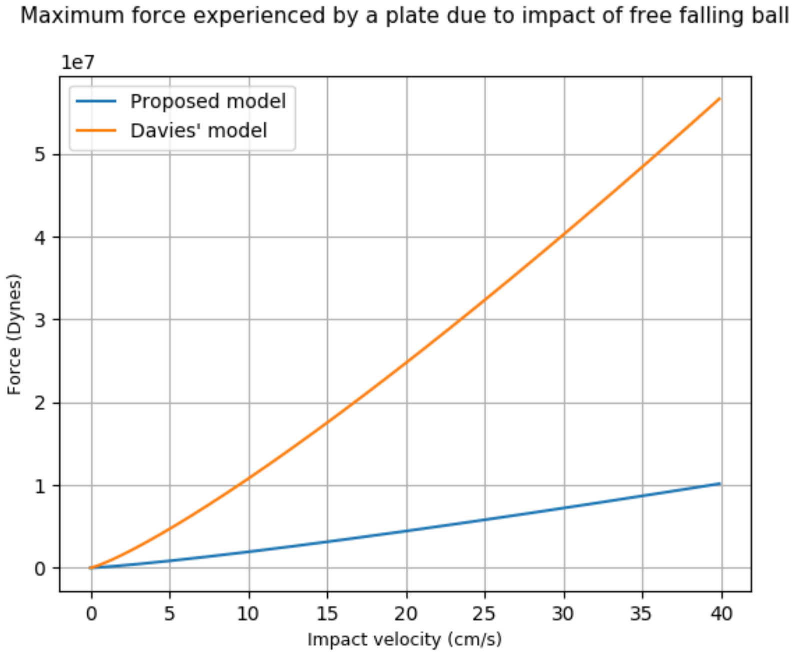

Figure 1. Comparison of the prediction of maximum normal force from proposed model with Davies' model assuming a ball of radius 1cm

In general, the proposed model predicts forces five times lower than those predicted by Davies' model. An immediate consequence of this, is the prediction of much lower maximum pressures, as shown above. Figure 5 shows the ratio of the maximum force predicted by Davies' model to that of the proposed model, and it is observed that a very consistent ratio is obtained for a significantly large range of impact velocity values.

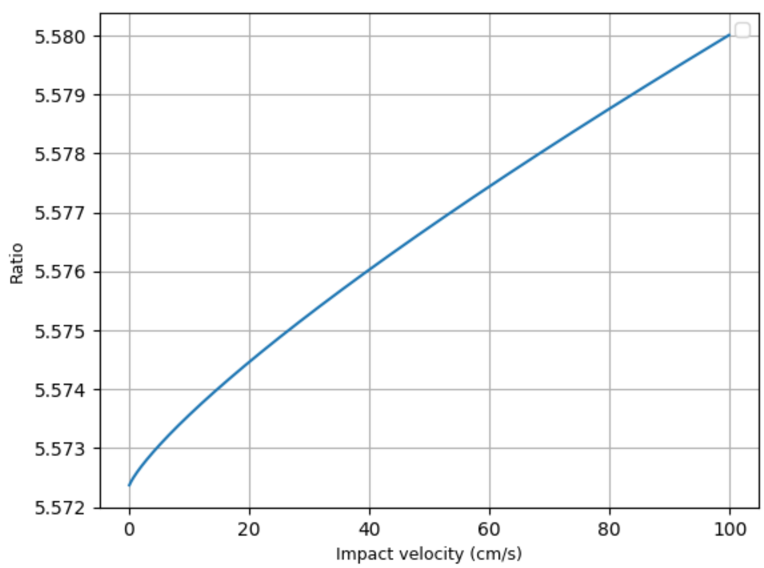

Figure 5. Ratio of predicted maximum force from Davies' model to proposed model

The consistency of this model, and the fact that it predicts lower forces and pressures, as it was sought, provides evidence of a change in the models being used to this day for the prediction of contact forces between moving objects.

The models proposed before have been used extensively without scrutinizing their validity, reason for which it is highlighted the need for an update of the models to be used in future.

6. Conclusion

In order to prevent material failure, it is crucial to know the loads tolerated by the material and the possible mechanisms of failure. In the literature, very few researches show the derivations for theoretical prediction of load magnitudes in dynamic scenarios. Most of the literature show results derived in the 1940s', without scrutinizing the validity of such models. In this research, the authors derive new equations for contact mechanics in a dynamic case. In particular, it was possible to derive an expression that predicts the maximum load and pressure experienced by a plate due to a ball hitting its surface after free fall. It was also possible to compare the proposed model with the model mostly used in the literature, fulfilling in this way the objectives of this research. It was observed that the proposed model predicts much lower maximum forces and pressures, being in average five times lower than Davies' contact model, which has been found to estimate surprisingly high values for pressures and forces in dynamic scenarios. These findings represent a new approach to calculate the contact forces and pressures; an aspect that has been abandoned to date.

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Biographies

Carlos Garcia-Mendoza is currently a Graduate Teaching Assistant (GTA) and Researcher at The Technical University of Manabi (UTM), Portoviejo, Ecuador. He received an M.Sc. in Mathematics from The Technical University of Manabi and a B.E. in Mechanical and Materials Engineering from The University of Queensland, Australia. His research interests in engineering include: material's performance, wear, mining equipment for wear estimation, etc. His research interests in mathematics include: topology, algebra and optimization. He has publications both in engineering and mathematics, some of them targeted to the improvement of the methods used to predict the material's performance and the systematization of experimental procedures. https://orcid.org/0000-0002-6580-2560

Neyfe Sablón Cossío is a Professor and investigator in career Industrial Engineering at the technical University of Manabí, Portoviejo, Ecuador. She holds a Bachelor's degree in Industrial Engineering from Matanzas University. She is also graduated with a Master degree in Business Administration and PhD in Science Technical Industrial Engineering. All these studies carried out in Cuba. She has published several journals and conference papers. Dra. Sablón has accomplished research project on supply chains in Mexico, Cuba and Ecuador. She holds the National Prize for the result of Scientific Research of the Academy of Sciences of Cuba. Her research interests include administrations, business, logistic, operations administrations, supply and value chain. https://orcid.org/0000-0002-6691-0037