Optimizing of Series systems with identical components and imperfect Repairs

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Abstract

The objective of this research is to determine the optimal solution to a series repairable system subjected to random failure. Any failed system can be repaired in one of three categories: perfect, minimal, and imperfect. In this article, we address the problem of systems undergoing imperfect repair after each failure. In addition to optimal system availability and minimum cost, closed analytical expressions for the various parameters are derived.

Key words: optimization, availability, Kuhn Tucker conditions, series systems, imperfect repair,

1 Introduction

Buildings, machines, roads, cars, and power plants, among other human-made structures, are prone to malfunction and failure, demanding maintenance and repair in order to function optimally or acceptably. Maintenance optimization is a term that refers to mathematical models that are used to discover the optimal balance between decreasing maintenance costs and maximizing benefits. The literature has looked into the optimization of mechanical systems availability is a result of not just dependability, but also maintainability and supportability.

In this paper, we look at the best behaviour of two variants of a series of repairable systems with identical components, assuming that the failure rate increases with each random failure. Another assumption is that repair times and times between failures are exponentially distributed, that travel time from and to the repair station is insignificant, and that placing a replacement component into operation takes time.

1.1 Objective

The study's major objective is to develop and an optimization model for a series repairable system with identical components that is imperfectly repaired. Second, derive the ideal parameter expressions, such as repair and failure rates, as well as a closed expression for the optimal solution.

2.1 Literature Review

Based on exponentially distributional failure and repair time system, Wolde and Ghobbar (2013) examined inspection problems associated with railway carrier. Historical failure and repair data were jointly used is in order to find the optimal interval inspection with minimum costs and maximum availability. Li et al (2018) inspected the viability of the maintenance scheme used by a multicomponent production system. The system is assumed to deteriorate over time and ultimately fail. The objective was to find the most cost-effective maintenance policy. Meanwhile; Hajipour and Taghipour (2016) proposed a model for two types of multicomponent repairable (hard-type and soft-type components) systems with the objective of finding the optimal no periodic inspection interval.

Volf (2021) was concerned with maximizing a technical device's preventative maintenance program. The main objective is to develop a model for determining the device's optimal life time while minimizing maintenance expenses. Lin et al. (2022) designed a multicomponent model optimization reliability model under s-dependent competing risks. The goal is to find the best reliability threshold as well as the best number of PM cycles at the lowest total predicted maintenance cost rate. Zhao et al. (2021) examined the performance of a cold standby system with two components that are subjected to δ -shocks and imperfect repairs and had varied reliability characteristics. Under shocks and incomplete repairs, geometric process models are used to characterize the lifetime and repair time. To quantify the system's economic performance, an equation for the long-run cost per unit time is also generated. For a two-unit series system. Zhang et al (2022) developed a combination optimization problem of condition-based maintenance policy and buffer capacity for a two-unit series system in their work. The semi-Markov process was used to examine the probabilities of the system's transition states under imperfect repairs. Moreover, an example was presented using simulation example to illustrate the superiority of the proposed joint optimization strategy

Barlow (1965) used the Markov process to examine system availability and also presented the principles of mathematical theory of reliability. More studies have concentrated on system instantaneous availability modeling since then, but steady-state availability is more relevant in engineering practice, where it was an important indicator to balance costs and benefits, such as military and space systems (Ibrahim (2015); Hajeeh, (2012)). (Sarkar and Sarkar, (2001); Biswas and Sarkar, (2003) used a recursive method to investigate the instantaneous and steady-state availability of an inspection-based system, with lifetime and repair time following discrete functions

Cui and Xie (2005) assumed that frequent inspections occur at predetermined intervals following repairs or replacements in the event of failures. Under the premise of random repair or replacement time, some general conclusions on immediate and steady-state availability were reported. Furthermore, Li et al. (2008) investigated the availability of a periodic inspection system with arbitrary lifetime and repair time distributions and the link between the inspection duration and availability under a perfect repair policy. It should be highlighted that all of the above results were achieved on the assumption that preventative maintenance (PM) time causes minimal downtime. Tang et al. (2004) looked into the availability of a regularly examined system, taking into account non-negligible downtime.

Ebeling [15] proposed that steady-state availability is influenced by system MTBF and MTTR, as well as maintenance and supply delay times, where MTBF is an important indicator of system reliability, MTTR is an important indicator of system maintainability, and maintenance and supply delay times reflect the ability of the support system.

3. Methods

We try to identify the best parameters for a series system with numerous components that go through n failures before complete regeneration in this paper. An optimization model is created; the model is a non-linear programming problem with inequality constraints, and the general formula is:

$$\text{Minimise} \quad z = f(\mathbf{X}) \qquad (1)$$
$$\text{Subject to} \quad \mathbf{g(X)} \geq 0$$

The Khun Tucker criteria will be utilized to determine the optimal solution, in this case, where the set of inequality constraints, $\mathbf{g(X)} \geq 0$ can be converted to equalities by using the proper nonnegative slack variables. Then, to satisfy the non-negativity conditions, Let $S_i^2\ (\geq 0)$ be the square of the slack variable subtracted from the i^{th} constraint.
Next, Define,
$\mathbf{S} = (S_1, S_2, \ldots, S_{\hat{I}})^T$ and $\mathbf{S}^2 = (S_1^2, S_2^2, \ldots, S_{\hat{I}}^2)^T$
be the slack variable vector and its corresponding square values vector, respectively. The Lagrangean function is given by:

$$\tilde{L}(\mathbf{X}, \mathbf{S}, \bar{\xi}) = f(\mathbf{X}) - \bar{\xi}\left[\mathbf{g(X)} - \mathbf{S^2}\right] \qquad (2)$$

where $\bar{\xi}$ represents the constraints vector $\mathbf{g(X)} \geq \mathbf{0}$. of Lagrange multipliers. The following is produced by taking the partial derivatives $\tilde{L}$ of with respect to $\mathbf{X}, \mathbf{S}, \bar{\xi}$ and setting them to zeros: Depending on

$$\frac{\partial \tilde{L}}{\partial \mathbf{X}} = \nabla f(\mathbf{X}) - \bar{\xi}\ \nabla \mathbf{g(X)} = \mathbf{0} \qquad (3)$$
$$\frac{\partial \tilde{L}}{\partial S_i} = 2\xi_i S_i = 0 \quad i = 1, 2, \ldots\ldots, \hat{I}$$
$$\frac{\partial \tilde{L}}{\partial \bar{\xi}} = -\left[\mathbf{g(X)} - \mathbf{S}^2\right] = \mathbf{0}$$

The following is derived from the above set of equations

$$\xi_i g_i(\mathbf{X}) = 0 \qquad i = 1, 2, \ldots, \hat{I} \qquad (4)$$

$$\nabla f(\mathbf{X}) - \bar{\xi}\ \nabla \mathbf{g(X)} = \mathbf{0} \qquad (5)$$
$$\xi_i g_i(\mathbf{X}) = 0 \qquad i = 1, 2, \ldots, \hat{I}$$
$$\mathbf{g(X)} \geq \mathbf{0}$$
$$\bar{\xi} \geq 0$$

The system under study is a single component series system undergoing imperfect were the system's function deteriorates after each failure and thus its failure rate increases. The system regenerates after the n^{th} failure. Detailed are exhibited in figure 1.

3.1 Case 1: perfect Repair

The optimization model for case l is as follows:

$$Minimize\ \ z = f(\lambda,\mu) = \frac{C_1}{\lambda} + C_2\mu + C_3\left(\frac{\lambda}{\lambda+\mu}\right) \tag{6}$$

$$\text{A} = \left(\frac{\mu}{\lambda+\mu}\right) \geq \rho_1$$

Subject To

$$\mu - \lambda \geq 0$$
$$\lambda \geq 0$$
$$\mu \geq 0$$

Where

C_1: The initial purchase per component, it is associated with the failure rate of the system.
C_2: The repair cost.
C_3: The steady state down time cost.
ρ_1: A given minimum desired level of availability.
λ: Failure rate, μ: Repair rate.
ρ_1: A given minimum desired level of availability.

$z = f(\lambda,\mu)$ = the objective function

The Lagranean cost function is as follows:

$$Minimise\ \ z = f(\lambda,\mu) = \frac{C_1}{\lambda} + C_2\mu + C_3\left(\frac{\lambda}{\lambda+\mu}\right) - \xi_1\left(\frac{\mu}{\lambda+\mu} - \rho_1\right) \tag{7}$$

Where:

ξ_i : Lagrange multiplier for the i^{th} inequality constraint.

The Khun Tucker conditions for the above optimisation problem is:

$$\frac{\partial f}{\partial \lambda} = -\frac{C_1}{\lambda^2} + C_3\left(\frac{\mu}{(\lambda+\mu)^2}\right) + \xi_1\left(\frac{\mu}{(\lambda+\mu)^2}\right) = 0 \tag{8}$$

$$\frac{\partial f}{\partial \mu} = C_2 - C_3\left(\frac{\lambda}{(\lambda+\mu)^2}\right) - \xi_1\left(\frac{\lambda}{(\lambda+\mu)^2}\right) = 0 \tag{9}$$

$$\xi_1\left(\frac{\mu}{(\lambda+\mu)} - \rho_1\right) = 0 \tag{10}$$

Therefore, from (10) since ς_1 is not 0, then $\left(\frac{\mu}{(\lambda+\mu)} - \rho_1\right) = 0 \Rightarrow \frac{\mu}{(\lambda+\mu)} = \rho_1$, which can be expressed as: $\mu(1-\rho_1)=\lambda\rho_1$ and hence $\mu=\lambda\rho$, Substituting $\lambda\rho$ for μ in (9) and (10), the following equations are obtained:

$$-\frac{C_1}{\lambda^2}+C_3\left(\frac{\rho}{\lambda(1+\rho)^2}\right)+\xi_1\left(\frac{\rho}{\lambda(1+\rho)^2}\right)=0 \tag{11}$$

$$C_2-C_3\left(\frac{1}{\lambda(1+\rho)^2}\right)-\xi_1\left(\frac{1}{\lambda(1+\rho)^2}\right)=0 \tag{12}$$

where $\rho=\frac{\rho_1}{(1-\rho_1)}$.

Multiplying (12) by ρ and adding to (11), λ^* is found, and by substitution, the optimal solution is as follows:

$$\lambda^*=\sqrt{\frac{C_1}{C_2\rho}},\ \mu^*=\sqrt{\frac{C_1\rho}{C_2}},\ \xi_1^*=(1+\rho)^2\sqrt{\frac{C_1C_2}{\rho}}-C_3$$

$$;\ z^*=2\sqrt{C_1C_2\rho}+\frac{C_3}{(1+\rho)}$$

Where ξ_1 id the Lagrange multipliers λ^*, μ^*,ξ^*, and z^* are the optimal values for the failure rate, repair rate, Lagrangian multiplier, and objective function, respectively.

3.2. Case 2: imperfect Repair

The system in question is a single component series system undergoing imperfect repair, with each failure deteriorating the system's function and increasing the failure rate. Figure 1 shows how the system regenerates after the nth failure.

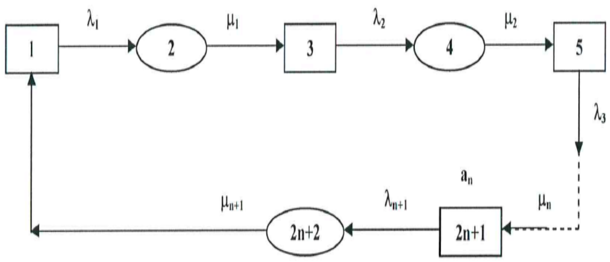

Fig. A single component repairable system under imperfect repair

The objective function in this situation has the same structure as in case 1. The steady state availability of the system depicted in the diagram is as follows: (Hajeeh, 2012)

$$A=\frac{\sum_{i=1}^{n+1}\frac{1}{\lambda_i}}{\left[\sum_{i=1}^{n+1}\left(\frac{1}{\lambda_i}+\frac{1}{\mu_i}\right)\right]} \tag{13}$$

The objective function structure is as follows:

$$f(\lambda,\mu)=\frac{C_1}{\lambda}+C_2\mu+C_3\left[\frac{\frac{\beta}{\mu}}{\frac{\alpha}{\lambda}+\frac{\beta}{\mu}}\right]-\xi_1\left[\frac{\frac{\alpha}{\lambda}}{\frac{\alpha}{\lambda}+\frac{\beta}{\mu}}-\rho_1\right] \tag{14}$$

All the variables as previously defined. Using the Khun Tucker conditions, the following equations are obtained:

$$\frac{\partial f}{\partial \lambda}=-\frac{C_1}{\lambda^2}+C_3\left(\frac{\rho}{\lambda(1+\rho)^2}\right)+\xi_1\left(\frac{\rho}{\lambda(1+\rho)^2}\right)=0 \tag{15}$$

$$\frac{\partial f}{\partial \mu}=C_2-C_3\left(\frac{\alpha}{\lambda\beta(1+\rho)^2}\right)-\xi_1\left(\frac{\alpha}{\lambda\beta(1+\rho)^2}\right)=0 \tag{15}$$

Solving the above set of equation in (14) and (15) as perfumed previously in perfect case, it can be deduced that the optimal solution for replacement at the $(n+1)$ th failure of one-component system with n repairs is as follows:

$$\lambda^*=\sqrt{\frac{C_1\alpha}{C_2\rho\beta}},\ \mu^*=\sqrt{\frac{C_1\beta\rho}{C_2\alpha}},\ \xi_1^*=(1+\rho)^2\sqrt{\frac{C_1C_2\beta}{\alpha\rho}}-C_3;$$
$$z^*=2\sqrt{\frac{C_1C_2\beta\rho}{\alpha}}+\frac{C_3}{(1+\rho)} \tag{16}$$

4. Numerical Example

A corporation purchased a machine to make particular types of aluminum profiles for \$10,000. The manager is seeking high performance and is interested in the entire cost associated with various levels of performance. The labor cost is and downtime cost are \$200 and \$500, respectively. The machine is subjected to failure and repair according to the exponential distribution. Table 1 provide the total cost, failure rate, repair rate for different system's availability with α =5 and β= 15.

Table. 1. Cost, availability for various failure rate, repair rates

Availability (ρ_1)	Total Cost (\$)	Failure Rate (λ)	Repair Rate (μ)
0.75	2953.427	7.071068	7.071068
0.8	3365.986	6.123724	8.164966
0.85	3962.301	5.144958	9.718253
0.9	4948.979	4.082483	12.24745
0.95	7143.052	2.809757	17.79513
0.99	16253.08	1.230915	40.62019
0.999	51614.45	0.387492	129.0349

5. Results and Discussion

The relationship between availability and cost is seen in Figure 2. It is clear that as availability increases, so does the cost. But, if the availability is less than 0.9, the cost remains unaffected considerably; however, if the availability is greater than 0.9, the cost rises more sharply.

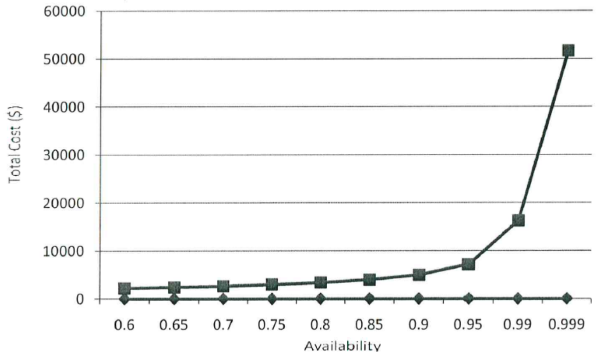

Fig.2. Total cost for various system's availability.

Meanwhile, Figure 3 exhibits the behavior of total system's cost as the failure rate (λ) increases. As λ increases the cost decreases. The cost is very high, at very small λ and declines as λ increases. At high values of λ, the cost decrees but a smaller rate

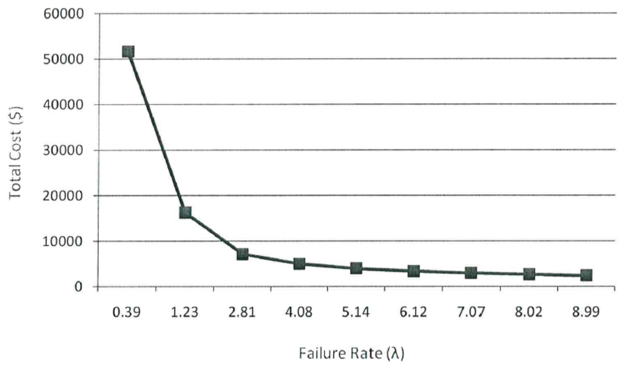

Fig.3. Total cost for different system's failure rates

6. Conclusion and Future Directions

Availability, reliability, and maintainability are key performance indicators for determining the fitness and resilience of operational systems. These characteristics are concurrently important as the cost. The expressions of the optimal parameters for a single repair system with imperfect repaid have been derived in this research effort. These expressions are generic and can be used event in perfect repair situation. The overall cost of a reparable system is a function of failure rate, repair rate, and availability, as these represent constituents of the cost function. In the analysis, it is observed that at lower levels of availability, the total cost of a single component system changes very minimally, but it surges at values of 0.90 and beyond. Contrastingly, the cost is highest at low failure rates and decreases as the failure rate rises.

We recommend that future research ought to investigate standby, and parallel systems as well as systems with other distributions.

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