Effect of Sample Rate on Frequency Values in Sinusoidal Signals

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Abstract

Analog signals are generally expressed by sinusoidal waves, where sinusoidal waves are the basis for all forms of analog signals. The important parameters of the sinusoidal signal equation are frequency (f_n) and sample rate (f_s) which are formulated with the equation $x(t) = \sin 2\pi f_n t$. The time parameter in the t variable in the sinusoidal equation is the duration of data collection for the sinusoidal signal. Frequency data in this study uses two models, namely the first model with a frequency value of $f_1 = 0.25$Hz, $f_2 = 1.5$Hz, $f_3 = 3$Hz and the second model with a frequency value of $f_1 = 0.25$Hz, $f_2 = 2$Hz, $f_3 = 4.25$Hz. The sample rate (f_s) used for both models is 8 Hz with a total time duration of 32 seconds. The method used in transforming the time domain of a sinusoidal signal into a frequency domain is using the Discrete Fourier Transform. The results obtained for the first model, namely the reading of the frequency value of $f_1 = 0.25$Hz, $f_2 = 1.5$Hz, $f_3 = 3$Hz can still be read clearly. As for the second model, the frequency value of $f_1 = 0.25$Hz, $f_2 = 2$Hz, $f_3 = 4.25$Hz can still be read. However, a new frequency appears between the frequency readings of $f_2 = 2$Hz and $f_3 = 4.25$ Hz, with a frequency value of $f = 3.75$Hz. This is because the use of the sample rate ($f_s = 8$ Hz) is smaller than 2 times the signal frequency ($2 f_n$). Therefore, the Nyquist theory in the case study of the second model does not meet the equation $f_s > 2 f_n$.

Keywords
Frequency, Sample Rate, Nyquist Theory.

1. Introduction
The important parameters of the sinusoidal signal equation are frequency (f_n) and sample rate (f_s) with the formulation $x(t) = \sin 2\pi f_n t$. The time parameter in the t variable in the sinusoidal equation is the duration of data collection for the sinusoidal signal. A sinusoidal signal is a function of time is a continuous-time signal. The continuous-time signal is converted into a discrete-time signal in performing signal processing. The sinusoidal signal in the time domain will be transformed into the frequency domain. Fourier transform is one of the methods that are often used in transforming the time domain to the frequency domain [Smith, 2003; Taylor, 2003].

The sample rate (f_s) needs to be determined in carrying out the transformation from the time domain to the frequency domain. The sample rate value used must be greater than the highest frequency value of the sinusoidal signal as shown in Figure 1. In determining the sample rate (f_s) the variables that need to be considered in signal processing are the Nyquist frequency and the Nyquist level [Hathaway, 2005].
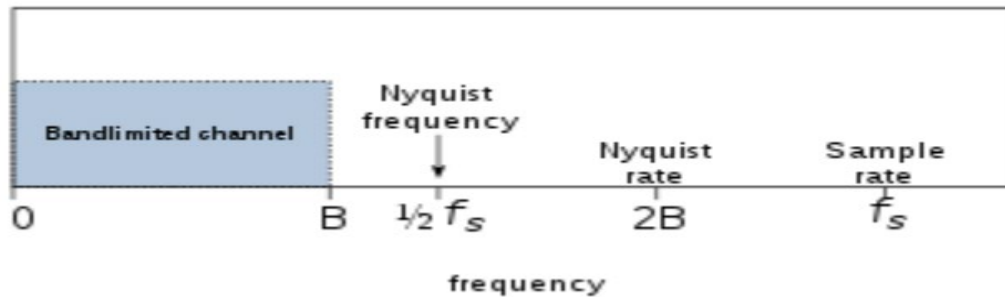

Figure 1. Relationship of Frequency and Nyquist Rate

In this study, we want to maintain the highest frequency value of the sinusoidal signal. This study will use two frequency models, namely f1 = 0.25 Hz, f2 = 1.5 Hz, f3 = 3 Hz, and a sinusoidal signal with a frequency of f1 = 0.25 Hz, f2 = 2 Hz, f3 = 4.25 Hz. The sample rate used for both models is fs=8 Hz. The analysis was conducted to examine whether the reading of the highest frequency value can still be read with the application of the specified sample rate (fs) value.

2. Literature Review

The Nyquist frequency is a characteristic of the sampler which converts a continuous signal function into a discrete data series. The value of the Nyquist frequency is half of the sampling rate in discrete signal processing in cycles per second or Hertz [Ulf, 1959; Stiltz, 1979]. If the domain of the function is time, the sample rate is usually expressed in samples per second, and the unit of Nyquist frequency is cycles per second (Hertz). When the function domain is distance, as in an image sampling system, the sample rate may be dots per inch and the corresponding Nyquist frequency will be in cycles per inch [Condon, 2016; Leis, 2011].

When the highest frequency (bandwidth) of a signal is less than the Nyquist or sampler frequency, the resulting discrete-time sequence is said to be free of distortion or aliasing, and the sample rate is appropriate because it is above the Nyquist level. In a sampling application, the first thing to do is to select the highest frequency to maintain. This selection is based on the accuracy and the expected frequency value of a sinusoidal signal. After assigning a high-frequency value to a signal, the next step is to determine an anti-aliasing filter in front of the sampler.

The job of the anti-aliasing filter is to attenuate frequencies above the frequency limit. Finally, based on the characteristics of the filter, it is possible to determine a sample rate that will give an acceptable small amount of aliasing (distortion). Aliasing is a frequency accumulation event in the signal spectrum because the signal contains components with a frequency of more than half the sampling frequency that has been sampled, resulting in damage to the information contained in the information signal [Lai, 2003; Fadali, 2020].

3. Methods

The work steps in determining the sample rate (f_s) in processing the signal from the time domain to the frequency domain are as shown in Figure 2. The signal data used is a sinusoidal function in time duration. The sample rate used in the study is fs = 8 Hz. The frequency value (f_n) has been determined and calculated into the equation $x(t) = \sin 2\pi f_n t$. Sinusoidal signals are made into 2 models, namely sinusoidal signals with a frequency of f_1 = 0.25 Hz, f_2 = 1.5 Hz, f_3 = 3 Hz, and a sinusoidal signal with a frequency of f_1 = 0.25 Hz, f_2 = 2 Hz, f_3 = 4.25 Hz.

The two sinusoidal signal equations, x(t) is transformed into the frequency domain using Fourier transform analysis. This transformation analysis aims to obtain the frequency value of the two sinusoidal signal data. If the frequency value has been determined at the beginning (model 1 with a frequency of f_1 = 0.25 Hz, f_2 = 1.5 Hz, f_3 = 3 Hz and model 2 with a frequency of f_1 = 0.25 Hz, f_2 = 2 Hz, f_3 = 4.25 Hz) the value is the same as the reading of the structure frequency, then the sample rate (f_s) used is appropriate. If the reading of the frequency value of the results of the

Fourier Transform analysis of the frequency value is different, or there is an additional new frequency value, then the sample rate (f_s) used needs to be increased in value.
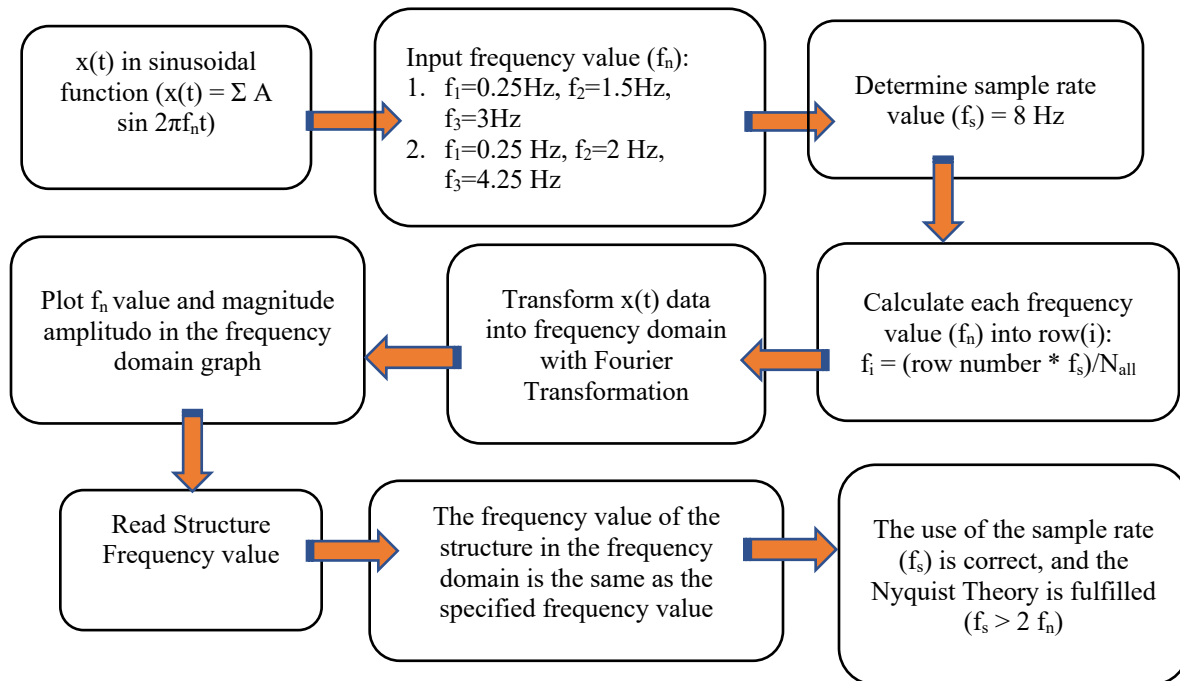
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┌──────────────────┐     ┌──────────────────────┐     ┌──────────────────┐
│  x(t) in         │     │ Input frequency       │     │                  │
│  sinusoidal      │ ──▶ │ value (fₙ):           │ ──▶ │ Determine sample │
│  function (x(t) = │     │ 1. f₁=0.25Hz,         │     │ rate value       │
│  Σ A sin 2πfₙt)  │     │    f₂=1.5Hz, f₃=3Hz   │     │ (fₛ) = 8 Hz      │
│                  │     │ 2. f₁=0.25 Hz,        │     │                  │
│                  │     │    f₂=2 Hz, f₃=4.25 Hz│     │                  │
└──────────────────┘     └──────────────────────┘     └──────────────────┘

┌──────────────────┐     ┌──────────────────┐     ┌──────────────────────┐
│ Plot fₙ value    │     │ Transform x(t)   │     │ Calculate each        │
│ and magnitude    │ ◀── │ data into        │ ◀── │ frequency value (fₙ)  │
│ amplitudo in the │     │ frequency domain │     │ into row(i):          │
│ frequency domain │     │ with Fourier     │     │ fᵢ = (row number *    │
│ graph            │     │ Transformation   │     │ fₛ)/Nₐₗₗ             │
└──────────────────┘     └──────────────────┘     └──────────────────────┘

┌──────────────────┐     ┌──────────────────────┐     ┌──────────────────────┐
│ Read Structure   │     │ The frequency value  │     │ The use of the sample │
│ Frequency value  │ ──▶ │ of the structure in  │ ──▶ │ rate (fₛ) is correct, │
│                  │     │ the frequency domain │     │ and the Nyquist       │
│                  │     │ is the same as the   │     │ Theory is fulfilled   │
│                  │     │ specified frequency  │     │ (fₛ > 2 fₙ)           │
│                  │     │ value                │     │                       │
└──────────────────┘     └──────────────────────┘     └──────────────────────┘
```

Figure 2. Flowchart of Determination of Sample Rate (f_s) Sinusoidal Signal

4. Results and Discussion

4.1 Calculation of the Determined Frequency Value of the Sinusoidal Function
   ($f_1 = 0.25$ Hz, $f_2 = 1.5$ Hz, $f_3 = 3$ Hz)

Given a signal with a sinusoidal function $x(t) = \sin 2\pi f_n t$, where f_n is the value of frequency and t in time units. The sinusoidal signal uses three frequency values, namely the first frequency (f_1) = 0.25 Hz, the second frequency (f_2) = 1.5 Hz, and the third frequency (f_3) = 3 Hz. The sample rate (f_s) and the amount of data (N) used are 8 Hz and 256. Threfore the total time duration for analyzing sinusoidal signals is 32 seconds. The vibration pattern of a sinusoidal signal with input data as described in Figure 3.
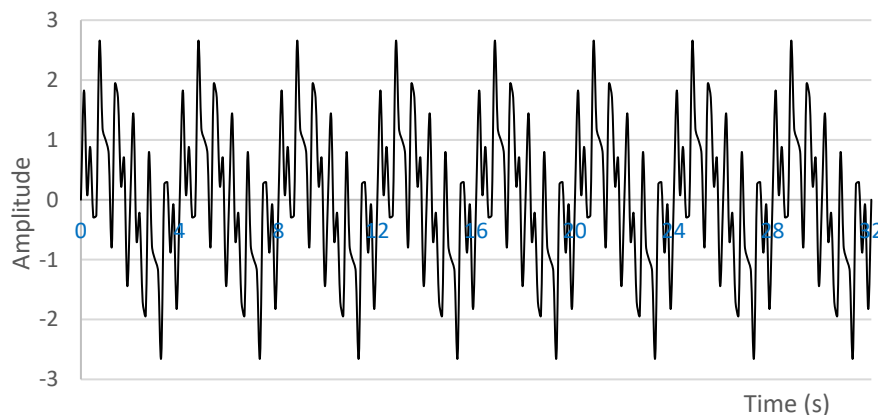

Figure 3. Sinusoidal Signal in Function of Time, x(t)

The sinusoidal signal in the function of time x(t) will be transformed into the frequency domain using the Discrete Fourier Transform. This discrete use will convert continuous signal data with a value of t seconds into discrete data (n_i). Discrete data is a series of data series that will start from n = 0.1,2,3…., N-1, where N depends on the amount of data used in the Discrete Fourier Transform analysis. The following is a method of transforming discrete data into the frequency domain using the Discrete Fourier Transform as shown in Figure 4.

Discrete Data, x(n)	Fourier Transfromation Discrete →	Frequency Domain $X(\omega_k)$
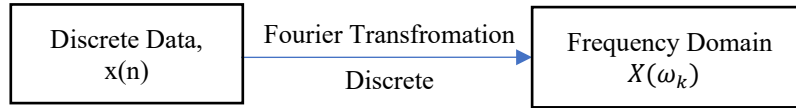
Figure 4. Discrete Data Transformation to Frequency Domain

Calculations using the Discrete Fourier Transform according to equation 1 with the following formula

$$X(\omega_k) = \sum_n^{N-1} x(n)e^{\frac{-j2\pi kn}{N}}, \quad k = 0,1,2,\ldots\ldots,N-1 \tag{1}$$

The amount of data used in the analysis is N=256, so there will be 256 discrete data points at x(n). Equation 2 by entering data N=256 will be as follows

$$X(\omega_k) = \sum_0^{255} x(n)e^{\frac{-j2\pi kn}{N}}, \quad k = 0,1,2,\ldots\ldots,255 \tag{2}$$

k is a discrete point in the frequency domain starting from $k = 0,1,2,\ldots\ldots,255$ for N=256. Threfore for the equation in the frequency domain ($X(\omega_k)$) is the sum of $x(n)e^{\frac{-j2\pi kn}{N}}$, where n starts from n=0 to n=255. The formula for the frequency domain $X(\omega_k)$ with the case of a sinusoidal signal with N=256 is as follows:

$$X(\omega_k) = x(n_0)e^{\frac{-j2\pi.k.0}{256}} + x(n_1)e^{\frac{-j2\pi.k.1}{256}} + x(n_2)e^{\frac{-j2\pi.k.2}{256}} + \cdots + x(n_{255})e^{\frac{-j2\pi.k.255}{256}} \tag{3}$$

The next step is to input the value of $k = 0,1,2,\ldots\ldots,255$.

- For the value of k=0 so that it will be obtained on the 0th line (beginning) in the frequency domain, the value of is obtained $X(\omega_0)$, with the following equation.
  $$X(\omega_0) = x(n_0)e^{\frac{-j2\pi.0.0}{256}} + x(n_1)e^{\frac{-j2\pi.0.1}{256}} + \cdots + x(n_{255})e^{\frac{-j2\pi.0.255}{256}}$$
  $X(\omega_0)$ value can also be written with the following equation because of the value of $e^0=1$.
  $$X(\omega_0) = x(n_0) + x(n_1) + \cdots + x(n_{255})$$
  By inputting the value $x(n_0)$ up to $x(n_{255})$ then the value $X(\omega_0)$ will be obtained.
  $$X(\omega_0) = 0 + 1{,}8261 + \cdots - 1.8261 = 0$$
- For k = 2, 4, 5, ……255 calculation of value $X(\omega_k)$ will be as shown in Table 1
- For the value of k=8 so that it will be obtained in the 8th row in the frequency domain, the value of is obtained $X(\omega_8)$, with the following equation.
  $$X(\omega_8) = x(n_0)e^{\frac{-j2\pi.8.0}{256}} + x(n_1)e^{\frac{-j2\pi.8.1}{256}} + \cdots + x(n_{255})e^{\frac{-j2\pi.8.255}{256}}$$
  By inputting the value $x(n_0)$ up to $x(n_{255})$ then the value will be obtained $X(\omega_8)$.
  $$X(\omega_8) = 0.e^{\frac{-j2\pi.8.0}{256}} + 1{,}8261.e^{\frac{-j2\pi.8.1}{256}} + \cdots - 1{,}8261.e^{\frac{-j2\pi.8.255}{256}} = -128j$$
  The value -128j is a complex number value, so it needs to be converted to a real number by calculating
  $$IMABS(z) = |z| = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-128)^2} = 128$$

Table 1. Calculation of Discrete Fourier Transform Data N=256, Sample Frequency
f_s = 8 Hz and Frequency f_1 = 0.25 Hz, f_2 = 1.5 Hz, f_3 = 3 Hz

Discrete Data Row-n (n_i)	Row Data in time (t)	Frequency, f_n (Hz)			Discrete Data Value, x(n)	Frequency Data Row-k	Frequency Domain Value $X(\omega_k)$	Amplitude
		f_1	f_2	f_3				
0	0.000	0.25	1.5	3	0.0000	0.0000	0	0.00
1	0.125	0.25	1.5	3	1.8261	0.0313	0	0.00
2	0.250	0.25	1.5	3	0.0898	0.0625	0	0.00
3	0.375	0.25	1.5	3	0.8800	0.0938	0	0.00
4	0.500	0.25	1.5	3	-0.2929	0.1250	0	0.00
5	0.625	0.25	1.5	3	-0.2583	0.1563	0	0.00
6	0.750	0.25	1.5	3	2.6310	0.1875	0	0.00
7	0.875	0.25	1.5	3	1.1976	0.2188	0	0.00
8	1.000	0.25	1.5	3	1.0000	0.2500	-128i	128.00
9	1.125	0.25	1.5	3	0.7640	0.2813	0	0.00
10	1.250	0.25	1.5	3	-0.7832	0.3125	0	0.00
11	1.375	0.25	1.5	3	1.9213	0.3438	0	0.00
12	1.500	0.25	1.5	3	1.7071	0.3750	0	0.00
13	1.625	0.25	1.5	3	0.2311	0.4063	0	0.00
14	1.750	0.25	1.5	3	0.6756	0.4375	0	0.00
15	1.875	0.25	1.5	3	-1.4359	0.4688	0	0.00
16	2.000	0.25	1.5	3	0.0000	0.5000	0	0.00
17	2.125	0.25	1.5	3	1.4359	0.5313	0	0.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
251	31.375	0.25	1.5	3	0.2583	7.8438	0	0.00
252	31.500	0.25	1.5	3	0.2929	7.8750	0	0.00
253	31.625	0.25	1.5	3	-0.8800	7.9063	0	0.00
254	31.750	0.25	1.5	3	-0.0898	7.9375	0	0.00
255	31.875	0.25	1.5	3	-1.8261	7.9688	0	0.00

Based on Table 1, it is made in the form of a comparison graph between the frequency and the amplitude value as shown in Figure 5. The frequency value for each k^{th} row depends on the sample rate (f_s) and the amount of data (N) used. This calculation using the value of f_s = 8 Hz. So that can be determined the value of delta f (Δf) for each adjacent frequency line, namely the sample rate (f_s) compared to the amount of data (N) worth f = 8/256 = 0.03125.

From Figure 5, which is a graph with the frequency domain compared to the amplitude, can still be clearly read the frequency value of f_1 = 0.25 Hz, f_2 = 1.5 Hz, f_3 = 3 Hz. Figure 5 shows that using the data N=64 and the sample rate, fs = 8 Hz, and the frequency values f_1 = 0.25 Hz, f_2 = 1.5 Hz, f_3 = 3 Hz, we get the frequency values f_1, f_2, and f_3 which are the same when calculated using the Discrete Fourier Transform. This is because the use of the sample rate (f_s = 8 Hz) is still greater than 2 times the signal frequency (2 f_n). So that the Nyquist theory in this case study is still fulfilled, namely f_s > 2 f_n.

Figure 5 is a comparison of the amount of data N=256, N=128, and N=64 for the same sample rate and frequency, namely f_s=8 Hz and frequency f_1 = 0.25 Hz, f_2 = 1.5 Hz, f_3 = 3 Hz. The amount of data does not affect the value of the frequency because it produces the same frequency value for the use of the three N data amounts. However, the smaller

the number of N data used, the lower the amplitude value at the frequency point as shown in Figure 5. On the contrary, the use of N data amounts which more can clarify the reading of the frequency value because of the dominant amplitude peak at the frequency point.
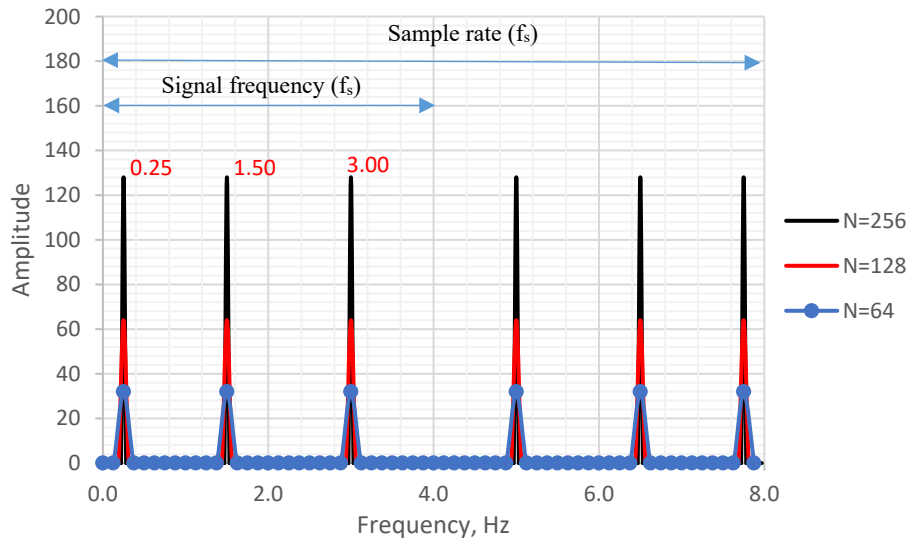

Figure 5. Frequency Graph with Data N=64, N=128 and N=256 Sample Rate f_s = 8 Hz and Frequency f_1 = 0.25 Hz, f_2 = 1.5 Hz, f_3 = 3 Hz

4.2 Calculation of the Determined Frequency Value of the Sinusoidal Function (f_1 = 0.25 Hz, f_2 = 2 Hz, f_3 = 4.25 Hz)

Based on Figure 6, the frequency values of f_1 = 0.25 Hz, f_2 = 2 Hz, f_3 = 4.25 Hz can still be read in the frequency domain with the dominant amplitude peaks at the frequency points of the structure. However, if you look back between the structural frequencies of f_2 = 2 Hz and f_3 = 4.25 Hz, a new dominant amplitude peak appears at the frequency of f = 3.75 Hz. The frequency value of f = 3.75 Hz is not a structure frequency value in the previously determined sinusoidal function signal, because in this sinusoidal function the structure frequency values have been determined, namely f_1 = 0.25 Hz, f_2 = 2 Hz, f_3 = 4.25 Hz.
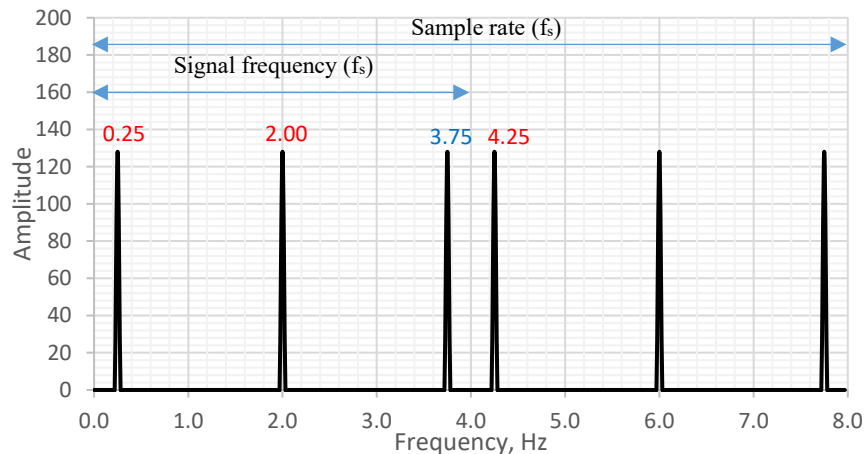

Figure 6. Frequency Graph with Data N=256, Sample Rate f_s = 8 Hz and Frequency f_1 = 0.25 Hz, f_2 = 2 Hz, f_3 = 4.25 Hz

According to Nyquist theory, the sample rate fs required in analyzing the signal is twice the value of the structure frequency ($f_s \geq 2\ f_n$). From this theory to the specified structure frequency value, there is one structural frequency value that does not meet the Nyquist theory rules, namely the frequency of the third vibration mode, namely $f_3 = 4.25$ Hz. The value of the sample rate f_s required in analyzing the signal is a minimum of 2 times the structure frequency, fn. In this case, if in a signal there is a structure frequency value of 4.25 Hz, then a sample rate value of at least 2 x 4.25 Hz is required or equivalent to $f_s = 8.5$ Hz.

The use of a sample rate that is less than 2 times the structure frequency causes the appearance of amplitude peaks that are not the structure frequency. Generally, the peak of the amplitude will appear between the two frequencies of the structure, or commonly referred to as the shadow frequency. The existence of an image frequency in a frequency domain is called aliasing, namely the presence of a dominant amplitude peak value at frequency points with the same value or similar to the structure frequency. This will have an impact on difficulties and doubts in reading the frequency value of the structure on the frequency domain graph.

5. Conclusion

The conclusion of this study is shown as follows:
- The use of sample rate (f_s) is an influential factor in transforming data from discrete-time domain to frequency domain using Discrete Fourier Transform.
- The sample rate (f_s) used must be 2 times the value of the structure frequency, to get the real value of the structure frequency of the discrete-time signal.
- The number of N data does not affect the frequency value. The smaller the number of N data used in data analysis, the amplitude value at the frequency points decreases.

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Biography

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