A New Formulation of Quadratic Assignment Problem (QAP)

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Abstract

Different formulations of QAP (such as Linear Integer Formulation, MILP formulation, Formulation by Permutations, Trace Formulation and Graph Formulation) are given in Loiola et. al. (2007). Different linearization of QAP are summarized in Singh and Sharma (2008). We give here a new QAP formulation and describe its advantages. For many different contributions to layout problems refer to Sharma (2019, 2019, 2020, 2020, 2021 and 2022). In this paper we give a new formulation of QAP.

Problem QAP
Min sum over (i,j,k,l): x(i,j)*x(k,l)*D(j,l)*F(i,k) (a)
Sum(i), x(i,j) = 1 for all j (b)
Sum(j), x(i,j) = 1 for all i (c)
x(i,j) = (0,1) for all i and j (d)

x(i,j) = 1 if facility 'i' gets into 'j' th slot, D(j,l) is the distance between 'j' th and 'l' th slot and F(i,k) is flow between facility 'i' and 'k'.

Here we give a new formulation of QAP. We put following constraints.
x(i,j) + M*(1 – y(i,j)) >= 1 for all i,j (e)
x(i,j) - M*(y(i,j)) <= 0 for all i,j (f)
y(i,j) = (0,1) for all i,j (g)
and add (h) in place of (d)
x(i,j) >= 0 for all i,j (h)

when x(i,j) is 1, then y(i,j) = 1; and when x(i,j) is 0, then y(i,j) is 0. Any x(i,j) in open interval (0,1) is infeasible (say x(i,j) = 0.5 is infeasible with either y(i,j) = 0 or 1). Thus though x(I,j) is real >= 0, it acts like binary due to (e), (f) and (g).

Then new formulation (NF_QAP) of QAP is
(a), s.t. (b), (c), (e), (f), (g) and (h).

It has N^2 real, N^2 binary and 2*(N^2 +N) constraints; and has quadratic objective function in real variables. These are the merits of our new formulation of QAP. For a detailed comparison see Singh and Sharma (2008). Similar formulations are possible in DPLP (dynamic plant layout problem) also.

Keywords
Quadratic Assignment Problem, New Formulation of QAP, Simple Plant Layout Problem, Linearization of QAP, DPLP (dynamic plant layout problem).

1. Introduction
The simple plant layout problem (SPLP or the quadratic assignment problem QAP) is posed as follows. There are N slots where N facilities are to be located. Distance between slot 'j' and 'l' is known and denoted by D(j,l). The material flow between machines (facilities) are known in advance (it is determine by product mix of the company and technological requirements of sequence in which different operations are performed on different jobs). We seek to minimize the material handling cost of the total shop when all machines are considered. It results in quadratic objective function involving binary variables s.t. linear constraints. It is a single period problem. When material flow between

machines changes (due to change of product mix of the company in different time periods) a multi-period problem results when machines are dismantled from one location and placed in other slot. This multi-period problem is referred to as dynamic plant layout problem (DPLP).

It is well known that problem SPLP is NP-Hard and computationally intractable. Hence heuristics are popularly used to solve QAP. These are Genetic Algorithms, Tabu Search, Simulated Annealing etc. For a detailed exposition refer to Singh and Sharma (2006). Researchers also have tried to linearize the objective function by adding more real and/or binary variables. A summary formulations of different formulations are given in Singh and Sharma (2008) and reproduced in this paper in appendix.

In this paper we give a new formulation of QAP that has N^2 binary and real variables and results in quadratic objective function in real variables. It has by far least number of binary and real variables. It is given in section 2. In section 3 we give a comparison table of different linearizations that appear in literature. Finally in section 4 we give our conclusions.

2. Contribution Details

We start with linear programming relaxation of QAP.

LP relaxation of QAP is:

Problem LPR_QAP

Min sum over (i,j,k,l): x(i,j)*x(k,l)*D(j,l)*F(i,k) (1)

Sum(i), x(i,j) = 1 for all j (2)
Sum(j), x(i,j) = 1 for all i (3)

x(i,j) >= 0 for all i,j (4)

If in problem LPR_QAP we put (5) in place of (4), then we get problem QAP.

x(i,j) = (0,1) for all i and j (5)

Thus the problem **QAP** is:

(1), s.t. (2), (3) and (5).

we define y(i,j) = (0,1) and define a new problem P

Problem P

Min sum over (i,j,k,l): x(i,j)*x(k,l)*D(j,l)*F(i,k)

Sum(i), x(i,j) = 1 for all j
Sum(j), x(i,j) = 1 for all i

x(i,j) + M*(1 – y(i,j)) >= 0.85 for all i,j (6)
x(i,j) - M*(y(i,j)) <= 0.15 for all i,j (7)

y(i,j) = (0,1) (8)

and (4).

If x(i,j) = 0.9, then y(i,j) = 1 and constraints (6) an (7) are satisfied and x(i,j) = 0.9 is a feasible solution to problem P. If x(i,j) = 0.1, then y(i,j) = 0 and both equations (6) and (7) are satisfied; and x(i,j) = 0.1 is a feasible solution to problem P. However: when x(i,j) = 0.6, then even when y(i,j) = 0 or 1, it remains infeasible. Thus it can be seen that x(i,j) in the open interval (0.15,0.85) falls in infeasible range.

If in (6) instead of 0.85 we write 1 and instead of 0.15 we write 0 in (7), that is,

x(i,j) + M*(1 – y(i,j)) >= 1 for all i,j (9)
x(i,j) - M*(y(i,j)) <= 0 for all i,j (10)

then x(i,j) >= 0 and real acts like a binary variable.

Using ideas above, we have a new formulation of QAP that is given below.

Problem NF_QAP

Min (1); s.t. (2), (3), (4), (8), (9) and (10).

So then we have in new formulation of QAP that has quadratic objective function in real variables only, and has additional y(i,j) binary variables that appear only in constraints. In the table given below we give number of binary variables, real variables and constraints in different formulations of QAP that appeared in literature (detailed formulations appear in the appendix).

3. A Comparison

Comparison of Sizes of Different Formulations and Linearization of QAP is presented in Table 1.

Table 1. Comparison of Sizes of Different Formulations and Linearization of QAP

Different Formulations of QAP that appeared in literature	No. of Binary Variables	No. of Real Variables	No. of Constraints
Lawler (1962)	N^3	-	N^4
Kauffman and Broecks (1978)	N^2	N^2	$N^2 + 2*N$
Christofides (1980)	$N^4 + N^2$	-	$N^4 + 2*N + 1$
Frieze and Yadegar (1983)	N^2	N^4	$N^4 + 2*N^3 + 2*N^2$
Asad and Xu (1985)	$N^4 + N^2$	-	$3*N^2$
Padberg and Rijal (1996)	$N^4 + N^2$		$3*N^3$
Singh and Sharma (2008)	N^2	N^4	N^4
In the current paper	N^2	N^2	$2*N^2 + 2*N$

4. Conclusion

Intuitively it appears this formulation of QAP has advantages. In a numerical study by Singh and Sharma (2008) it was reported that CMT (2008), SS (2008) and KB (1978) are overall better performers than all other linearizations. SS (2008) was found to be superior to CMT (2008) and statistically significant at 5% significance level (as revealed by a meta analysis). A numerical investigation is underway to see if new formulation of QAP (NF_QAP) has any significant advantages over the best linealizations of QAP.

APPENDIX
Lawler (1962):

Put y(i,j,k,l) = x(i,j)*x(k,l) and we

Min sum over (i,j,k,l): y(i,j,k,l)*D(j,l)*F(i,k) (11)

s.t. sum(i,j,k,l), y(i,j,k,l) = n^2 (12)

x(i,j) + x(k,l) – 2*y(i,j,k,l) >= 0 (13)

y(i,j,k,l) = (0,1) (14)

(L): (11), s.t. (2), (3), (5), (12), (13) and (14)

Kauffman and Broecks (1978)

Min y(i,j) (15)

Sum(k,l), F(i,k)*D(j,l)*x(i,j) + Sum(k,l), F(i,k)*D(j,l)*x(k,l) –
y(i,j) <= Sum(k,l), F(i,k)*D(j,l) for all i,j (16)

y(i,j) >= 0 (17)

(KB): (15), s.t. (2), (3), (5), (16), (17)

Christofides (1980):

y(i,j,k,l) >= x(i,j) + x(k,l) – 1 (18)

(CMT): (11), s.t. (2), (3), (5), (14) and (18)

As noted by Singh and Sharma (2008), in CMT if (14) is replaced by (19) as given below, it still remains a valid formulation of QAP.

y(i,j,k,l) >= 0 (19)

Frieze and Yadegar (1983)

Sum(i), y(i,j,k,l) = x(j,l) for all k,j,l (20)

Sum(k), y(i,j,k,l) = x(j,l) for all i,j,l (21)

Sum(j), y(i,j,k,l) = x(i,k) for all i,k,l (22)

Sum(l), y(i,j,k,l) = x(i,k) for all i,k,j (23)

y(i,i,k,k) = x(i,k) for all i,k (24)

(FY): (11), s.t. (2), (3), (5), (19), and (20) - (24)

Assad and Xu (1985)

Sum(j,l), y(i,j,k,l) = n*x(j,l) for all j,l (25)

Sum(k), y(i,j,k,l) = x(i,k) for all i,k,l (26 a)

sum(l), y(i,j,k,l) = x(i,k) for all i,j,k (26 b)

(AX): (11), s.t. (2), (3), (5), (19), and (25) and (26)

Padberg and Rijal (1996)

Here x(i,j) = 1 means facility i goes to slot j; and x(k,l) = 1 means that facility k goes to slot l; and y(i,j,k,l) = x(i,j)*x(k,l); and following holds.

-x(i,j) + sum(k=1 to i-1), y(k,l,i,j) + sum(k,i+1 to n), y(i,j,k,l) = 0   for i, j   (27)

-x(i,j) + sum(l=1 to j-1), y(k,l,i,j) + sum(l,j+1 to n), y(k,l,i,j) = 0   for k<i, j   (28)

-x(i,j) + sum(l=1 to j-1), y(i,j,k,l) + sum(l,j+1 to n), y(i,j,k,l) = 0   for i<k and j (30)

(PR): (11), s.t. (2), (3), (5), (19), (22) and (27) and (28)

Singh and Sharma (2008)

y(i,j,k,l) – BiG_Positive_Number*(x(i,j)+x(k,l) – 2) >= 1   (31)

(SS): (11), s.t. (2), (3), (5), (19) and (31)

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Biography

Prof. RRK Sharma: He is B.E. (mechanical engineering) from VNIT Nagpur India, and PhD in management from I.I.M., Ahmedabad, INDIA. He has nearly three years of experience in automotive companies in India (Tata Motors and TVS-Suzuki). He has 33 years of teaching and research experience at the Department of Industrial and Management Engineering, I.I.T., Kanpur, 208016 INDIA. To date he has written 1217 papers (peer-reviewed (401) /under review (35) / working papers 781 (not referred)). He has developed over ten software products. To date, he has guided 66 M TECH and 23 Ph D theses at I.I.T. Kanpur. He has been Sanjay Mittal Chair Professor at IIT KANPUR

(15.09.2015 to 14.09.2018) and is currently a H.A.G. scale professor at I.I.T. Kanpur. In 2015, he received "Membership Award" given by IABE USA (International Academy of Business and Economics). In 2016 he received the "Distinguished Educator Award" from IEOM (Industrial Engineering and Operations Management) Society, U.S.A. In 2021, he received IEOM Distinguished Service Award. In 2019, 2020, 2021 and 2022 he was invited by the Ministry of Human Resources Department, India, to participate in the NIRF rankings survey for management schools in India. In 2019, 2020 and 2022 he was invited to participate in the Q.S. ranking exercise for ranking management schools in South Asia.