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Optimal Lot Sizing and Setup Cost Reduction for Deteriorating Production System

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Abstract

In this paper, we extend Porteus (1986) to generalize the economic production quantity (EPQ) model with finite production rate and to derive an exact expression for the expected total annual cost function when maintenance actions and capital investment in setup cost reduction are adopted. In addition, the uniqueness property of the optimal lot size and the bounds of optimal lot size are derived whether or not the investment is made for setup cost reduction. Furthermore, a numerical example reveal that Porteus's solution procedure may cause significant penalty costs sometimes such that validities of Theorems as those proposed in Porteus (1986) are questionable. So, the main purpose of this paper is not only to remove shortcomings but also to present an exact derivation of the expected total annual cost function and provide an appropriate solution algorithm to find the optimal lot size and setup cost.

Keywords
Lot sizing, setup reduction, deteriorating production system.

1. Introduction

In the classical inventory model, it is implicitly assumed that all the items produced are of perfect quality and quality level is fixed at an optimal level. However, in real production environment, we can often observe that the product quality is not always perfect and usually depends on the state of the production process. In addition, it can often be observed that there are defective items being produced due to imperfect production process. The defective items can be handled in various ways, e.g. to be rejected, repaired, reworked, or, if they have reached the customer, refunded. In all cases, substantial costs are incurred. Hence, the inventory policy determined by the conventional model with perfect quality may be inappropriate. In order to study the effects of imperfect quality to lot sizes, several authors have developed various models involving the quality-related issues. For example, Rosenblatt and Lee (1986) and Porteus (1986) are among the first who explicitly elaborated on a significant relationship between quality imperfection and lot size, and show that it is better to produce lot size smaller than the classical EPQ. Specifically, Porteus (1986) considers an EOQ model and assumes that all items produced are defective when the system is out-of-control. However, Rosenblatt and Lee (1986) assume that a proportion of the items produced are defective once the production process is out-of-control. Recently, Freimer et al. (2006) extend the work of Rosenblatt and Lee (1986) by considering that the fraction of items produced is a general increasing function of time, rather than depending on a specific distribution of the time until the process goes to out-of-control. More recently, Maddah et al. (2010) extend the work of Porteus (1986) by assuming that imperfect quality items may not be reworked, and hence will have to be discarded and removed from inventory at the end of a lot production. Other related studies on lot sizing involving the quality-related issues include Cheng (1991), Khouja and Mehrez (1994), Makis (1998), Salameh and Jaber (2000), Cardenas-Barron (2000), Goyal and Cardenas-Barron (2002), Chang (2004), Hou (2005) Papachristos and Konstantaras (2006), Chiu et al. (2007), Wee et al. (2007), Maddah and Jaber (2008), Chang and Ho (2010), Khan et al. (2011), and their references.

On the other hand, the benefits of reduced setups are well documented. For example, faster changeovers have been associated with lower inventory, faster throughput, shorter lead time, improved quality, and lower unit cost.

Quick setups are also considered an important element for successfully implementing just-in-time production or time-based competition. Much of the analytical work in setup reduction examines the benefits of reduced setups in inventory and setup costs. Therefore, considerable attention has been paid to the optimal lot sizing and optimal setup cost investment strategies. In the literature, Porteus (1985) first introduced the concept and developed a framework for investing in reducing setup cost in the EOQ model. Porteus (1986) further studied the effects of process quality improvement and setup cost reduction on the lot size by introducing the investing options in the EOQ model. Porteus' work has encouraged many researchers to examine the relationships between the amount of capital investment and setup cost level (e.g. Keller and Noori, 1988, Kim et al. (1992), Paknejad et al. (1995), Hofmann (1998), Ouyang and Chang (2000), Lin and Hou (2005), and Hou (2007)).

However, for a mathematical simplicity, Porteus (1986) use a truncated Taylor series expansion for the exponential term to present the total annual cost function. Consequently, he can only derive an approximated solution which is not an optimal one and results in overvalue of total annual cost whether or not investment is made for setup cost reduction. Hence, his solution procedure may be inappropriately to obtain the optimal lot size and investment for setup cost reduction when based on the actual cost function of analytical model as proposed in this paper. Consequently, the main purpose of this paper is threefold:

(1) An economic production quantity (EPQ) model with a finite production rate and a Markov production process is developed. Based on exact model, we show that there exists a unique optimal lot size such that the expected total annual cost is minimized.
(2) The paper derives the closed forms for the upper and lower bounds on the optimal lot size when or no investment is made for setup cost reduction, thereby the appropriate search procedures are provided whether or not investment is made for setup cost reduction.
(3) This paper compares optimal solutions obtained by using our approach and Porteus approach. Numerical examples show that our approach is better whether or not investment is made for setup cost reduction. In particular, the considerable cost savings could be realized through the setup cost reduction when we used in the exact model instead of approximate model.

2. Assumptions and notations

Consider a production system where a single item is produced on a single machine. Suppose that the operating condition of the production system at any time can be classified into one of two states, i.e., in-control and out-of-control. When the system is in the in-control state, it may either shift to the out-of-control state with probability q or stay in the in-control state with probability 1-q during the production of an item. Once the system shifts to the out-of-control state, it remains in this state until the end of a production run. That is, the deteriorating process follows a two-state discrete-time Markov chain during production of a lot with a transition occurring with each unit produced. After the production of a lot of size Q, the production process is inspected to reveal the state of the process. If the process is out-of-control then it is restored back to the in-control state with maintenance or adjustment cost R for the production of the next lot. However, it is not reasonable to be considered in Porteus (1986) on analyzing the effect of production process on the optimal lot size. Therefore, the process can be assumed to be in-control state before beginning production of the lot. Due to manufacturing variability, if the production process is out-of-control then it will lead to an item produced is defective with probability θ not rather than all items produced are defective as in Porteus (1986).The defective items will eventually be reworked with cost c_r such that the capacity of the production system is not affected.

In addition, the relationship between setup cost reduction and capital investment can be described by the logarithmic investment cost function. That is, setup cost, K, and the capital investment in setup cost reduction, ϕ_k, can be stated as

$$\phi_k(K) = b\ln\left(\frac{K_0}{K}\right) \quad \text{for } 0 < K \le K_0 \tag{1}$$

where 1/b is the fraction of the reduction in K per dollar increase in investment. This investment function is similar those of Porteus (1986), and it has also been widely used in literature, see, e.g. Keller and Noori (1988), Hong and Hayya (1995), and Hou (2007).

To develop an EPQ model for the production system as mentioned above, the following notations are needed and those in Porteus (1986) are also used in this paper.

m = demand rate,

p = production rate, $p > m$,

K_0 = original setup cost for each production run prior to investment,
K = setup cost for each production run,
h = holding cost per unit per unit time,
c_r = rework cost for a defective item,
R= maintenance cost for restoring the process from out-of-control back to in-control,
q = the probability that the system from in-control state shifts to out-of-control state,
$\bar{q}$ = the probability that the system stays in-control state during the production of an item and $\bar{q} = 1 - q$,
θ = the percentage of defective items produced when the process is in the out-of-control state,
Q = lot size for each production run,
N= number of defective items produced in a lot of size Q
$EAC(Q)$= expected total annual cost for a lot of size Q,
ϕ_k = capital investment in setup cost reduction,
i = the fractional per unit time opportunity cost of capital.

3. Mathematical Model

The objective of the analysis in this section is to determine the optimal lot size Q^* and setup cost K^* such that the expected total annual cost is minimized when maintenance and reworking actions are taken into account. The following results are derived.

Lemma 1. The expected number of defective items in a lot of size Q is

$$E(N) = \theta(Q - \sum_{j=1}^{Q} \bar{q}^j) \tag{2}$$

Given that the lot size is Q, the expected total annual cost incurred in a production cycle includes setup cost, inventory holding cost, maintenance cost and rework cost which are derived as follows.

(1) Setup costs:
The setup cost in a production cycle is K.

(2) Holding cost:
The inventory holding cost in a production cycle is $\frac{hQ^2(p-m)}{2pm}$.

(3) Maintenance cost:
The maintenance cost occurs only when the production process is out-of-control at the end of a production run for a lot of size Q. Hence, the expected maintenance cost becomes $R(1-\bar{q}^Q)$.

(4) Rework cost: According to Eq. (2), the expected total rework cost per cycle is given by

$$c_r E(N) = c_r\theta(Q - \sum_{j=1}^{Q} \bar{q}^j) \tag{3}$$

Clearly, the expected total cost of a production cycle becomes the sum of the setup cost, inventory holding cost, maintenance cost and rework cost, as follows:

$$TC(Q) = K_0 + \frac{hQ^2(p-m)}{2pm} + R(1-\bar{q}^Q) + c_r\theta(Q - \sum_{j=1}^{Q} \bar{q}^j) \tag{4}$$

Next, the time duration of a production cycle T is equal to Q/m. Then, the expected total cost per unit time is given by

$$f(Q) = TC(Q)/T = \frac{mK_0}{Q} + \frac{h(p-m)}{2p}Q + c_r m\theta + \frac{m}{Q}\left[R(1-\bar{q}^Q) - c_r\theta\sum_{j=1}^{Q}\bar{q}^j)\right] \tag{5}$$

As it takes investment to reduce setup cost, we should include an amortized investment cost in our proposal model. Therefore, the expected total annual cost of the system, $ETC(Q, K)$, is composed of Eq. (5) and the amortized total capital cost, $i\phi_k(K)$ which shows the economic consequences of the investment per unit time, as follows:

$$ETC(Q,K) = \frac{mK}{Q} + \frac{h(p-m)}{2p}Q + c_r m\theta + \frac{m}{Q}\left[R(1-\bar{q}^Q) - c_r\theta\sum_{j=1}^{Q}\bar{q}^j)\right] + i\phi_k(K) \tag{6}$$

where $\phi_k(K)$ is based on Eq. (1). Hence, Eq. (6) can be expressed as

$$ETC(Q,K) = \frac{mK}{Q} + \frac{h(p-m)}{2p}Q + c_r m\theta + \frac{m}{Q}\left[R(1-\bar{q}^{Q}) - c_r\theta\sum_{j=1}^{Q}\bar{q}^{j})\right] + ib\ln\left(\frac{K_0}{K}\right) \tag{7}$$

So far, we have appropriately established mathematical model that is particularly useful on investing in reduced setups in the EPQ model with an imperfect production process. The problem is then to derive the optimal lot size Q^* and setup cost K^* that leads to the minimum of $ETC(Q, K)$ in Eq.(7). In addition, based on Eq. (7), following observations should be mentioned:

(1) When $q = 0$, $\theta = 0$, and $K^* = K_0$, no capital investment in setup cost reduction is made as well as the production process does not deteriorate and system is always in the in-control state. It is explicitly that all items produced are perfect quality. Then, Eq. (7) reduces to the traditional EPQ model with perfect quality. In other words, the traditional EPQ model is a special case of the model proposed in this paper.

(2) Note that as p approaches to infinity, it is equivalent to instantaneous production of the entire lot, the Eq. (7) reduces to the EOQ model with an imperfect production process.

(3) When R =0, Eq. (7) reduces to the case that the maintenance cost is not considered.

(4) When $\theta = 1$ and combining the arguments of (2) and (3), Eq. (7) will reduce to the model in Porteus (1986). Therefore, the proposed model in this paper is also an extension of Porteus (1986).

It is easy to show that the $ETC(Q,K)$ in Eq. (7) is convex in K. In order to find the minimum cost, the partial derivatives of the $ETC(Q,K)$ can be evaluated as

$$\frac{\partial ETC(Q,K)}{\partial K} = \frac{m}{Q} - \frac{ib}{K} = 0. \tag{8}$$

From (8), the optimal setup cost can be easily solved as

$$K^*(Q) = ib\frac{Q}{m} \tag{9}$$

Substituting (9) into (7) yields the following expression of the corresponding expected total annual cost EAC:

$$EAC(Q) = ib + \frac{h(p-m)}{2p}Q + c_r m\theta + \frac{m}{Q}\left[R(1-\bar{q}^{Q}) - c_r\theta\sum_{j=1}^{Q}\bar{q}^{j})\right] + ib\ln\left(\frac{K_0 m}{ibQ}\right) \tag{10}$$

Differentiating (10) with respect to Q, we have

$$\frac{\mathrm{d}EAC(Q)}{\mathrm{d}Q} = \frac{h(p-m)}{2p} - \frac{m}{Q^2}\beta(1-\bar{q}^{Q} + Q\bar{q}^{Q}\ln\bar{q}) - \frac{ib}{Q}. \tag{11}$$

where $\beta = R - \frac{c_r\theta\bar{q}}{q}$

Let

$$z(Q) = Q\frac{\mathrm{d}EAC(Q)}{\mathrm{d}Q} \tag{12}$$

Then Q^* is the optimal solution of $EAC(Q)$ if and only if $z(Q^*)=0$.

Equation (12) yields

$$z(Q) = \frac{h(p-m)}{2p}Q - \frac{m}{Q}\beta(1-\bar{q}^{Q} + Q\bar{q}^{Q}\ln\bar{q}) - ib \tag{13}$$

In addition, we also have the following results.

Theorem 1. The optimal lot size Q^* exists and is unique.

Theorem 2. (a) If $\beta \le 0$ then $0 < Q^* \le Q_1^* \le Q_2^*$

(b) If $\beta > 0$ then $0 < Q_1^* < Q^* < Q_2^*$

where $Q_1^* = \frac{2pib}{h(p-m)}$ and $Q_2^* = pib + \sqrt{(2pib)^2 + 8h(p-m)pmR}$

Notice that Q_1^* is also the optimal solution for traditional EPQ model when setup cost can be reduced through capital investment. In addition, It is worth pointing out that optimal lot size Q^* is larger than the optimal solution of the traditional EPQ model in case of $\beta > 0$. It is unlike the results obtained in Porteus (1986). The reason is that he does not take the maintenance cost into account properly.

In addition, the optimal capital investment in setup cost reduction can be determined by following equation (14) when Q^* is obtained.

$$\phi_k^* = b\ln\left(\frac{K_0 m}{ibQ^*}\right). \tag{14}$$

Hence, $z(Q^*)=0$ has a unique optimal solution by Theorem 2, which implies the optimal lot size, Q^* minimizing $EAC(Q)$ always exists and is unique. Although there is no closed form expression for Q^*, it can be obtained through the use of numerical methods. Theorems 1 and 2 reveal that the bisection method based on the intermediate value theorem (see, eg, Varberg et al., 2007) is appropriately used to find Q^*. Basically, it is rather simple, practical, accurate and efficient. Hence, using the uniqueness property in Theorem 1 and the bounds in Theorem 2, the following algorithm 1 is appropriately used to find Q^*.

Algorithm 1:

Step 1: Find $Q^* \in [Q_1, Q_2]$ such that $z(Q^*) = 0$ by using the bisection method.

Step 2: Substitute Q^* into Eq. (9) to solve $K^*(Q^*)$.

Step 3: Compare $K^*(Q^*)$ and K_0. If $K^*(Q^*) \le K_0$, then $K_{opt} = K^*(Q^*)$ by using Eq. (9), and capital investment for setup cost reduction is obtained by using Eq. (14). Otherwise, set $K_{opt} = K_0$, and no capital investment for setup cost reduction is required.

Step 4: The point (Q^*, K_{opt}) is the optimal solution, and calculate the corresponding expected annual total cost $EAC(Q^*)$ utilizing Eq. (10).

4. Illustrative Example

Two different models used for comparison are specified as follows.
Model 1: lot size model with capital investment in setup cost reduction (see, Eq. (10)), and Model 2: lot size model without capital investment in setup cost reduction (see Eq. (5)).
The percentage of savings of expected total annual cost is defined by

$$\%EAC = \left[1 - \frac{EAC(Q^*)\text{ of model1}}{EAC(Q^*)\text{ of model2}}\right] \times 100\%$$

To illustrate the above procedure, let us consider the following numerical example:

Example 1: Let p = 1500 units/year, m = 1000 units/year, K_0 = \$100, h = \$8, c_r = \$5, R = \$25, and θ = 0.60. Besides, we take i = 0.15 and a = 390 when q = 0.01(reliable system) and 0.2(unreliable system). Using the above parameters, it can be verified that the values of β are β = -272 < 0 and β = 13 > 0 for q = 0.01 and q = 0.2, respectively. Applying the proposed procedure as mentioned above, we provide the results for the two models as in Table 1. From the results shown in Table 1, we see that very significant savings of the expected total annual cost are achieved through capital investment. Here, when the case of q = 0.01(reliable system), we show that the optimal capital investment (ϕ_k) require \$2540.696 when the optimal setup cost K^* = \$0.148 and the optimal lot size Q^* = 3.166 are required. In this case, the corresponding expected total annual cost $EAC(Q^*)$ = \$665.247. On the other hand, when the case of q = 0.2(unreliable system), we show that the optimal capital investment (ϕ_k) require \$ 1130.11 when the optimal setup cost K^* = \$5.515 and the optimal lot size Q^* = 117.84 are required. In this case, the corresponding expected total annual cost $EAC(Q^*)$ = \$ 3449.852. Both the two cases, the corresponding expected total annual cost, in particularly for q = 0.01 case, are more less than the cost when there is no capital investment for setup cost reduction. Hence, the percentages of expected total annual cost savings are given by 72.3% and 8.6% for q = 0.01 and q = 0.2, respectively. That is, 72.3% and 8.6% of the expected total annual cost savings for q = 0.01 and q = 0.2, respectively are relative to the model without capital investment in setup cost reduction.

Table 1. The results for Example1.

q values	Model type	Q^*	K^*	$EAC(Q^*)$	% EAC

			$(\phi_k(K^*))$		
q = 0.01(reliable system)	Model 1	3.166	0.148 (2540.696)	665.247	72.3
q = 0.2(unreliable system)	Model 1	117.84	5.515 (1130.11)	3449.852	8.6
q = 0.01(reliable system)	Model 2	111.121	100	2401.518	---
q = 0.2(unreliable system)	Model 2	291.119	100	3776.316	---

Note: %*EAC* is defined the percentage of cost savings in *EAC* as compared with Model 2.

5. Conclusions

Considering that the reduction of setup cost is an important strategy in manufacturing, the present investigations on setup cost in a manufacturing environment are warranted. In this paper, we develop the optimal lot sizing policies when setup cost can be reduced through capital investment. To explore these policies, the expected total annual cost function with capital investment is formulated. We show that the cost function is convex such that the optimal lot size Q^* exists and is unique. Furthermore, an effective algorithm is developed to find the optimal lot size and then setup cost. Therefore, the optimal capital investment is appropriately determined. However, for a mathematical simplicity, Porteus (1986) could only derive an approximated solution by a truncated Taylor series approximation which is not an optimal one and results in overvalue of the expected total annual cost whether or not investment is made for setup cost reduction. Hence, his solution procedure may be inappropriately to obtain the optimal lot size and investment for setup cost reduction when based on the actual cost function as proposed in this paper. Finally, a numerical example was given to illustrate the results and evaluate the effects of utilizing capital investments. It should be emphasized that these results show that significant cost savings can be achieved by adopting capital investment.

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References

Cárdenas-Barrón, L.E., 2000, Observation on: economic production quantity model for items with imperfect quality, International Journal of Production Economics, 67 (2), 201.

Chang, H.C., 2004, An application of fuzzy sets theory to the EOQ model with imperfect quality items, Computer & Operations Research, 31, 2079-2092.

Chang, H.C. and Ho, C.H., 2010, Exact closed-form solutions for optimal inventory model for items with imperfect quality and shortage backordering, Omega, 38, 233-237.

Cheng, T.C.E., 1991, EPQ with process capability and quality assurance considerations, Journal of Operational Research Society, 42, 713-720.

Chiu, S. W., Wang, S. L., and Peter Chiu, Y. S., 2007, Determining the optimal run time for EPO model with scrap, rework, and stochastic breakdowns, European Journal of Operational Research,180, 664-676.

Freimer, M., Thomas, D., Tyworth., J., 2006. The value of setup cost reduction and process improvement for the economic production quantity model with defects. European Journal of Operational Research, 173(1), 241-251.

Goyal, S.K. and Cárdenas-Barrón, L.E., 2002, Note on: economic production quantity model for items with imperfect quality-a practical approach, International Journal of Production Economics, 77, 85-87.

Hofmann, C., 1998, Investments in modern production technology and the cash flow-oriented EPQ-model, International Journal of Production Economics, 54, 193-206.

Hong, J. D., and Hayya, J. C., 1995, Joint investment in quality improvement and setup reduction, Computers Operations Research, 22, 567-574.

Hou, K.L., 2005, Optimal production run length for deteriorating production system with a two-state continuous-time Markovian processes under allowable shortages, Journal of the Operational Research Society, 56, 346-350.

Hou, K.L., 2007, An EPQ model with setup cost and process quality as functions of capital expenditure, Applied Mathematical Modelling, 31, 10-17.

Keller, G., and Noori, H., 1988, Justifying new technology acquisition through its impact on the cost of running an inventory policy. IIE Transitions, 20, 284-291.

Khan, M., Jaber, M.Y., and Bonney, M., 2011, An economic order quantity (EOQ) for items with imperfect quality and inspection errors, International Journal of Production Economics, 133, 113-118.

Khouja, M. and Mehrez, A., 1994, Economic production lot size model with variable production rate and imperfect quality, Journal of the Operational Research Society, **45**, 1405-1417.

Kim, K.L., Hayya, J.C., and Hong, J.D., 1992, Setup reduction in economic production quantity model, Decision science 23(2), 500-508.

Lin, L.C. and Hou, K.L., 2005, An inventory system with investment to reduce yield variability and set-up cost, Journal of the Operational Research Society, 56, 67-74.

Makis, V., 1998, Optimal lot sizing and inspection policy for EMQ model with imperfect inspections. Naval Research Logistics, **45**,165-186.

Maddah, B. and Jaber, M.Y., 2008, Economic order quantity for items with imperfect quantity: revisited, International Journal of Production Economics 112, 808-815.

Ouyang, L. Y., and Chang, 2000, Impact of investment in quality improvement on (Q, r, L) model involving the imperfect production process, Production Planning & Control, 11, 598-607.

Paknejad M. J., Nasri, F., and Affisco, J. F., 1995, Defective units in a continuous review (s,Q) system, International Journal of Production Research, **33**, 2767-2777.

Papachristors, S. and Konstantaras, I., 2006, Economic ordering quantity models for items with imperfect quality, International Journal of Production Economics, 100, 148-154.

Porteus, E.L., 1985, Investing in reduced setups in the EOQ model. Management Sciences, 31, 998-1010.

Porteus, E.L., 1986, Optimal lot sizing, process quality improvement and setup cost reduction, Operations Research, 34, 137-144.

Rosenblatt, M.J. and Lee, H.L., 1986, Economic production cycle with imperfect production processes, IIE Transitions, 18, 48-55.

Salameh, M.K. and Jaber, M.Y., 2000, Economic production quantity model for items with imperfect quality, International Journal of Production Economics, 64, 59-64.

Varberg, D., Purcell, E.J., and Rigdon, S.E., 2007, Calculus, ninth ed., Pearson Education Inc., N. J.

Wee, H.M., Yu, J., and Chen, M.C., 2007, Optimal inventory model for items with imperfect quality and shortage backordering, Omega, 35, 7-11.

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