Analysis of a Two-Echelon Inventory Model with Lost Sales and Stochastic Demand using Continuous-Time Markov chain

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Abstract

In this paper, we studied a two-echelon inventory system which consists of one supplier and one retailer. All installations follow base stock inventory management policy. The customers arrive at retailer according to Poisson distribution. The demand is satisfied when both supplier and retailer have positive on-hand inventory. Otherwise, the demand is lost. The lead time of supplier and the time needed to replenish orders of retailer are assumed to be exponentially distributed. The purpose of this study is to minimize the expected long-run total cost of the system per unit time. Total cost of the system includes holding costs of supplier and retailer, and the cost of lost sales. We modeled the problem using continuous-time Markov chain and provided an approximation of the steady-state joint probabilities of outstanding orders of retailer in the service of supplier and supplier's on-hand inventory. The optimal solutions, properties of the objective function as well as some numerical results are presented. In our numerical results, the maximum error of 2.85% was observed.

Keywords
Two-echelon inventory system, Base stock policy, Poisson demand, Lost sales

1. Introduction
This paper focuses on a two-echelon inventory system including one supplier and one retailer. Both supplier and retailer follow base stock inventory management policy with parameters *r* and *s*, respectively. The demands arrive to retailer according to a Poisson process with parameter λ. In case of no on-hand inventory at retailer, the demand is lost. Otherwise, retailer checks the on-hand inventory level of supplier; again, in case of no on-hand inventory at supplier, the demand is lost. If both retailer and supplier have positive on-hand inventory level, four events occur simultaneously: (A) the demand is satisfied, (B) as retailer follows base stock inventory policy, retailer demands one unit of product to supplier, (C) one unit of product joins the queue to be shipped to the retailer and (D) the supplier start to replenish its inventory. The lead time of supplier and the time needed to replenish orders of retailer are exponentially distributed with parameters μ and ν, respectively.

The purpose of this paper is to determine the optimal values of *r* and *s* which minimize the expected system total cost per unit time.

In the following sections, we formulate the problem using continuous-time Markov chain and will provide an approximation of the steady-state probabilities. Finally, we will discuss some properties of the objective function and will present some numerical examples.

2. Notations
The notation by which the system was modeled is provided in table 1.

Table 1: Model notation

Notation	Definition
λ	Demand rate
μ	Lead time rate
ν	Replenishment rate
r	Maximum inventory position of supplier
k	On-hand inventory of supplier, $0 \le k \le r$
s	Maximum inventory position of retailer
n	The number of outstanding orders of the retailer, that is, the queue of orders waiting to be shipped to the retailer, $0 \le n \le s$

h_1	Retailer holding cost per unit time per product
h_2	Supplier holding cost per unit time per product
π	Shortage cost per product
$\overline{I_1}$	Retailer long-run average on-hand inventory
$\overline{I_2}$	Supplier long-run average on-hand inventory
$\overline{S}$	Long-run average number of lost sales

3. Markov Model

The state space of the Markov chain model is (n,k), where n and k are defined in table 1. Let $\zeta(A)=1$ if A is true and $\zeta(A)=0$ otherwise. Therefore, the balance equations for the Markov chain model can be written as follows:

$$[\lambda\zeta(k\neq 0)\zeta(n\neq s)+\mu\zeta(n\neq 0)\zeta(k\neq 0)+(r-k)\nu]\pi(n,k)=$$

$$\mu\zeta(k\neq r)\zeta(n\neq s)\pi(n+1,k+1)+\lambda\zeta(n\neq 0)\zeta(k\neq 0)\pi(n-1,k)+(r-k+1)\nu\zeta(k\neq 0)\pi(n,k-1) \quad (1)$$

$$\sum_{k=0}^{r}\sum_{n=0}^{s}\pi(n,k)=1 \quad (2)$$

Let $\pi_1(n)$ ($0\leq n\leq s$) denote the probability of n customers (n outstanding orders) in the $M/M/1/s$ queue and let $\pi_2(k)$ ($0\leq k\leq r$) denote the probability of k free servers (k inventory) in $M/M/r/r$ queue. As in (Ross 2006), $\pi_1(n)$ and $\pi_2(k)$ can be calculated as below:

$$\pi_1(n)=\left(\frac{1-\rho_1}{1-\rho_1^{s+1}}\right)\rho_1^{\,n} \quad (3)$$

$$\pi_2(k)=\frac{\rho_2^{r-k}/(r-k)!}{\sum_{j=0}^{r}\rho_2^{j}/j!} \quad (4)$$

$$\rho_1=\frac{\lambda}{\mu} \quad (5)$$

$$\rho_2=\frac{\lambda}{\nu} \quad (6)$$

It can be shown that $\pi(n,k)=\pi_1(n)\pi_2(k)$ does not satisfy (1) and (2). However, we conjecture that the relation $\pi(n,k)=\pi_1(n)\pi_2(k)$ gives an acceptable approximation to our system steady-state probabilities. That is:

$$\pi(n,k)=\pi_1(n)\pi_2(k)=\left[\left(\frac{1-\rho_1}{1-\rho_1^{s+1}}\right)\rho_1^{\,n}\right]\left[\frac{\rho_2^{r-k}/(r-k)!}{\sum_{j=0}^{r}\rho_2^{j}/j!}\right],\quad 0\leq n\leq s,\quad 0\leq k\leq r \quad (7)$$

4. Objective Function

The objective function, denoted as $TC(r,s)$ is considered as the expected long-run cost of the system per unit time which is the summation of holding costs of supplier and retailer, and the cost of lost sales. Hence, the objective function is given by:

$$TC(r,s)=h_1\overline{I_1}+h_2\overline{I_2}+\pi\overline{S} \tag{8}$$

Theorem 1: $\overline{I_1}$, $\overline{I_2}$ and $\overline{S}$ are calculated as follows:

$$\overline{I_1}=\frac{1}{1-\rho_1}\left[\frac{\rho_1\left(\rho_1^s-1\right)-s\rho_1+s}{1-\rho_1^{s+1}}\right] \tag{9}$$

$$\overline{I_2}=r-\rho_2+\rho_2 B(r,\rho_2) \tag{10}$$

$$\overline{S}=\lambda B(r,\rho_2)\left[\frac{1-\rho_1{}^s}{1-\rho_1^{s+1}}\right]+\lambda\rho_1{}^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right] \tag{11}$$

Where $B(N,a)$ is Erlang loss function and is calculated as $B(N,a)=\frac{a^N}{N!}\Big/\sum_{j=0}^{N}\frac{a^j}{j!}$.

Proof: Using the steady-state probabilities presented in (7):

- **Calculating $\overline{I_1}$**

$$\overline{I_1}=\sum_{n=0}^{s}\sum_{k=0}^{r}(s-n)\pi(n,k)=\sum_{n=0}^{s}\sum_{k=0}^{r}(s-n)\left[\left(\frac{1-\rho_1}{1-\rho_1^{s+1}}\right)\rho_1{}^n\right]\left[\frac{\rho_2^{r-k}\big/(r-k)!}{\sum_{j=0}^{r}\rho_2^j\big/j!}\right]=\left(\frac{1-\rho_1}{1-\rho_1^{s+1}}\right)\sum_{n=0}^{s}(s-n)\rho_1^n$$

$$=\frac{1}{1-\rho_1}\left[\frac{\rho_1\left(\rho_1^s-1\right)-s\rho_1+s}{1-\rho_1^{s+1}}\right]$$

- **Calculating $\overline{I_2}$**

$$\overline{I_2}=\sum_{n=0}^{s}\sum_{k=0}^{r}k\pi(n,k)=\sum_{n=0}^{s}\sum_{k=0}^{r}k\left[\left(\frac{1-\rho_1}{1-\rho_1^{s+1}}\right)\rho_1{}^n\right]\left[\frac{\rho_2^{r-k}\big/(r-k)!}{\sum_{j=0}^{r}\rho_2^j\big/j!}\right]=\frac{1}{\sum_{j=0}^{r}\rho_2^j\big/j!}\sum_{k=0}^{r}k\rho_2^{r-k}\big/(r-k)!$$

$$=\frac{1}{\sum_{j=0}^{r}\rho_2^j\big/j!}\sum_{k=0}^{r}(r-k)\rho_2^k\big/k!=r-\rho_2+\rho_2 B(r,\rho_2)$$

- **Calculating $\overline{S}$**

$$\overline{S}=\lambda\left[\sum_{n=0}^{s}\pi(n,0)+\sum_{k=0}^{r}\pi(s,k)-\pi(s,0)\right]=\lambda\left[\sum_{n=0}^{s-1}\pi(n,0)+\sum_{k=0}^{r}\pi(s,k)\right]$$

$$=\lambda\left[\frac{\frac{1-\rho_1}{1-\rho_1^{s+1}}}{\sum_{j=0}^{r}\rho_2^j\big/j!}\right]\left[\frac{\rho_2^r}{r!}\sum_{n=0}^{s-1}\rho_1{}^n+\rho_1^s\sum_{k=0}^{r}\rho_2^{r-k}\big/(r-k)!\right]=\frac{\lambda\rho_2^r}{r!}\left[\frac{\frac{1-\rho_1}{1-\rho_1^{s+1}}}{\sum_{j=0}^{r}\rho_2^j\big/j!}\right]\left[\frac{1-\rho_1^s}{1-\rho_1}\right]+\lambda\rho_1^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right]$$

$$=\lambda\left[\frac{\rho_2^r\big/r!}{\sum_{j=0}^{r}\rho_2^j\big/j!}\right]\left[\frac{1-\rho_1^s}{1-\rho_1^{s+1}}\right]+\lambda\rho_1^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right]=\lambda B(r,\rho_2)\left[\frac{1-\rho_1^s}{1-\rho_1^{s+1}}\right]+\lambda\rho_1^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right]$$

Remark: Note that $\pi(s,0)$ is calculated twice in $\sum_{n=0}^{s}\pi(n,0)+\sum_{k=0}^{r}\pi(s,k)$. Hence, we subtracted $\pi(s,0)$ from $\sum_{n=0}^{s}\pi(n,0)+\sum_{k=0}^{r}\pi(s,k)$ in the calculation of $\overline{S}$.

5. Convexity Analysis

If $TC(r,s)$ is a convex function, then a local minimum of $TC(r,s)$ is its global minimum. Accordingly, we present the following theorem:

Theorem 2: The objective function $TC(r,s)$ is convex in terms of *r* and *s* for any $\rho_1 \in (0,1)$.

Proof: As in (8), the objective function is defined as $TC(r,s)=h_1\overline{I_1}+h_2\overline{I_2}+\pi\overline{S}$. If we show that each component of $TC(r,s)$ is convex in terms of *r* and *s*, then we can prove theorem 2.

5.1 Convexity Analysis of $h_1\overline{I_1}$

We define $C_1(s)$, $\Delta_1C_1(s)$ and $\Delta_2C_1(s)$ as follows:

$$C_1(s)=h_1\overline{I_1}=\frac{h_1}{1-\rho_1}\left[\frac{\rho_1(\rho_1^s-1)-s\rho_1+s}{1-\rho_1^{s+1}}\right] \tag{12}$$

$$\Delta_1C_1(s)=C_1(s+1)-C_1(s) \tag{13}$$

$$\Delta_2C_1(s)=\Delta_1C_1(s+1)-\Delta_1C_1(s) \tag{14}$$

With some algebraic manipulations one can show that:

$$\Delta_1C_1(s)=h_1\left[\frac{s\rho_1^{s+1}(-1+\rho_1)^2+\rho_1^{s+3}+2\rho_1^{s+1}-3\rho_1^{s+2}+\rho_1-1}{(-1+\rho_1)(-1+\rho_1^{s+2})(-1+\rho_1^{s+1})}\right] \tag{15}$$

$$\Delta_2C_1(s)=-\frac{h_1\rho_1^{s+1}(-1+\rho_1)\left[(\rho_1^{s+2}+1)(-1+\rho_1)s+\rho_1^{s+2}(-3+\rho_1)+3\rho_1-1\right]}{(-1+\rho_1^{s+3})(-1+\rho_1^{s+2})(-1+\rho_1^{s+1})} \tag{16}$$

Lemma 1: For all $s\in N$ and any $\rho_1\in(0,1)$, $(\rho_1^{s+2}+1)(-1+\rho_1)s+\rho_1^{s+2}(-3+\rho_1)+3\rho_1-1\prec 0$.

Proof: with some simplification we have:

$$(\rho_1^{s+2}+1)(-1+\rho_1)s+\rho_1^{s+2}(-3+\rho_1)+3\rho_1-1\prec 0 \Leftrightarrow (\rho_1^{s+2}+1)(s+1)\succ 2\rho_1\sum_{j=0}^{s}\rho_1^j \tag{17}$$

Lemma 2: For all $s \in N$ and any $\rho_1 \in (0,1)$, $\left(\rho_1^{s+2}+1\right)(s+1) \succ 2\rho_1 \sum_{j=0}^{s} \rho_1^j$.

Proof: We prove lemma 2 by induction on s. Let $P(s): \left(\rho_1^{s+2}+1\right)(s+1) \succ 2\rho_1 \sum_{j=0}^{s} \rho_1^j$. Firstly, we have to show that $P(1)$ is true:

$$P(1): \rho_1^3 + 1 \succ \rho_1(1+\rho_1) \Rightarrow (1+\rho_1)\left(\rho_1^2 - \rho_1 + 1\right) \succ \rho_1(1+\rho_1) \Rightarrow (\rho_1 - 1)^2 \succ 0$$

Which shows the statement $P(s)$ is true for $s=1$. Next, we have to show that if $P(k)$ holds then $P(k+1)$ also holds. We write $P(k)$ as follows:

$$P(k): \left(\rho_1^{k+2}+1\right)(k+1) \succ 2\rho_1 \sum_{j=0}^{k} \rho_1^j \tag{18}$$

We add $2\rho^{k+2}$ to both sides of $P(k)$. Therefore:

$$\left(\rho_1^{k+2}+1\right)(k+1) \succ 2\rho_1 \sum_{j=0}^{k} \rho_1^j \Rightarrow \left(\rho_1^{k+2}+1\right)(k+1) + 2\rho_1^{k+2} \succ 2\rho_1 \sum_{j=0}^{k+1} \rho_1^j$$

The proof of lemma 2 is complete by showing $\left(\rho_1^{k+2}+1\right)(k+1) + 2\rho_1^{k+2} \prec \left(\rho_1^{k+3}+1\right)(k+2)$:

$$\begin{aligned}
&\left(\rho_1^{k+2}+1\right)(k+1) + 2\rho_1^{k+2} \prec \left(\rho_1^{k+3}+1\right)(k+2) \\
&\Rightarrow \left(\rho_1^{k+2}+1\right)(k+2) - \left(\rho_1^{k+3}+1\right)(k+2) + \rho_1^{k+2} - 1 \prec 0 \\
&\Rightarrow (k+2)\left[\left(\rho_1^{k+2}+1\right) - \left(\rho_1^{k+3}+1\right)\right] + \rho_1^{k+2} - 1 \prec 0 \\
&\Rightarrow (k+2)\rho_1^{k+2}(1-\rho_1) + \rho_1^{k+2} - 1 \prec 0 \\
&\Rightarrow (k+2)\rho_1^{k+2} \prec 1 + \rho_1 + \cdots + \rho_1^{k+1} \\
&\Rightarrow \left(\rho_1^{k+2} - 1\right) + \left(\rho_1^{k+2} - \rho_1\right) + \cdots + \left(\rho_1^{k+2} - \rho_1^{k+1}\right) \prec 0
\end{aligned}$$

The statement is true, since each of its components is negative. The basis and the inductive step have been performed. As a result, $P(s)$ is true for all natural s and any $\rho_1 \in (0,1)$. With respect to lemma 1 and lemma 2, one can find that $\Delta_2 C_1(s) \succ 0$.

5.2 Convexity Analysis of $h_2 \overline{I_2}$

We define $C_2(r)$ as below:

$$C_2(r) = h_2 \overline{I_2} = h_2\left[r - \rho_2 + \rho_2 B(r,\rho_2)\right] = h_2(r - \rho_2) + h_2 \rho_2 B(r,\rho_2) \tag{19}$$

The first component of $C_2(r)$ is a linear function. Thus, it is a convex function. The second component is Erlang loss function multiplied by a positive coefficient. It is proved that Erlang loss function is a convex function (Jagers and Van Doorn 1986). Therefore, $C_2(r)$ is a convex function in terms of r.

5.3 Convexity Analysis of $\pi \overline{S}$

Before conducting the analysis, we make a simplification to $\pi \overline{S}$ as follows:

$$\pi \overline{S} = \pi\left[\lambda B(r,\rho_2)\left[\frac{1-\rho_1{}^s}{1-\rho_1^{s+1}}\right] + \lambda \rho_1{}^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right]\right] = \pi\lambda B(r,\rho_2)\left[\frac{1-\rho_1{}^s}{1-\rho_1^{s+1}}\right] + \pi\lambda\rho_1{}^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right]$$

Also, we define the following functions:

$$C_3(r,s) = \pi\lambda B(r,\rho_2)\left[\frac{1-\rho_1{}^s}{1-\rho_1^{s+1}}\right] \tag{20}$$

$$C_4(s) = \pi\lambda\rho_1{}^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right] \tag{21}$$

As $C_3(r,s)$ is in the product form of the two single-variable functions, namely $B(r,\rho_2)$ and $\frac{1-\rho_1{}^s}{1-\rho_1^{s+1}}$ it is easy to show that the second forward differences of $C_3(r,s)$, $\Delta_2 C_3(r,s)$ equals to:

$$\Delta_2 C_3(r,s) = \pi\lambda[B(r+1,\rho_2) - B(r,\rho_2)]\left[\frac{1-\rho_1{}^{s+1}}{1-\rho_1^{s+2}} - \frac{1-\rho_1{}^s}{1-\rho_1^{s+1}}\right] \tag{22}$$

The properties of convex functions are introduced in (Yüceer 2002). We must show that the second forward differences of $C_3(r,s)$ is negative. In (Jagerman 1974), it is shown that Erlang loss function $B(N,a)$ is a decreasing function of N, that is:

$$B(r+1,\rho_2) - B(r,\rho_2) \prec 0 \tag{23}$$

With some simple algebra, one can show that:

$$\frac{1-\rho_1{}^{s+1}}{1-\rho_1^{s+2}} - \frac{1-\rho_1{}^s}{1-\rho_1^{s+1}} \succ 0 \tag{24}$$

(23) and (24) imply that the second forward differences of $C_3(r,s)$ is negative. Hence, $C_3(r,s)$ is a convex function in terms of *r* and *s*.

We define $\Delta_1 C_4(s)$ and $\Delta_2 C_4(s)$ as below:

$$\Delta_1 C_4(s) = C_4(s+1) - C_4(s) = \pi\lambda(1-\rho_1)\left[\frac{\rho_1^s(\rho_1 - 1)}{(-1+\rho_1^{s+2})(-1+\rho_1^{s+1})}\right] \tag{25}$$

$$\Delta_2 C_4(s) = \Delta_1 C_4(s+1) - \Delta_1 C_4(s) = \pi\lambda\left[\frac{(-1+\rho_1)^3(\rho_1^{2s+2} + \rho_1^s)}{(-1+\rho_1^{s+3})(-1+\rho_1^{s+2})(-1+\rho_1^{s+1})}\right] \tag{26}$$

It is clear that $\Delta_2 C_4(s)$ is positive. Therefore, $C_4(s)$ is convex. As each component of $\pi\overline{S}$ is convex, it is concluded that $\pi\overline{S}$ is also convex.

Finally, the proof of theorem 2 is complete as the objective function components, namely $h_1\overline{I_1}$, $h_2\overline{I_2}$ and $\pi\overline{S}$ are convex functions of *r* and *s*.

6. Optimal Solutions

In the previous sections, we calculated $\overline{S} = \lambda\left[\sum_{n=0}^{s}\pi(n,0) + \sum_{k=0}^{r}\pi(s,k) - \pi(s,0)\right]$. Now, let $\tilde{S}$ be defined as $\tilde{S} = \lambda\left[\sum_{n=0}^{s}\pi(n,0) + \sum_{k=0}^{r}\pi(s,k)\right]$. Hence:

$$\tilde{S} = \lambda\left[\sum_{n=0}^{s}\pi(n,0)+\sum_{k=0}^{r}\pi(s,k)\right] = \lambda\left[\frac{\frac{1-\rho_1}{1-\rho_1^{s+1}}}{\sum_{j=0}^{r}\rho_2^j/j!}\right]\left[\frac{\rho_2^r}{r!}\sum_{n=0}^{s}\rho_1{}^n + \rho_1^s\sum_{k=0}^{r}\rho_2^{r-k}/(r-k)!\right]$$

$$= \frac{\lambda\rho_2^r}{r!}\left[\frac{\frac{1-\rho_1}{1-\rho_1^{s+1}}}{\sum_{j=0}^{r}\rho_2^j/j!}\right]\left[\frac{1-\rho_1^{s+1}}{1-\rho_1}\right] + \lambda\rho_1^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right] = \lambda\left[\frac{\rho_2^r/r!}{\sum_{j=0}^{r}\rho_2^j/j!}\right] + \lambda\rho_1^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right]$$

$$\Rightarrow \tilde{S} = \lambda B(r,\rho_2)+\lambda\rho_1^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right] \tag{27}$$

We define $\overline{TC}(r,s)$ as follows:

$$\overline{TC}(r,s) = h_1\overline{I_1} + h_2\overline{I_2} + \pi\tilde{S}$$

$$\Rightarrow \overline{TC}(r,s) = \frac{h_1}{1-\rho_1}\left[\frac{\rho_1(\rho_1^s-1)-s\rho_1+s}{1-\rho_1^{s+1}}\right] + h_2\left[r-\rho_2+\rho_2 B(r,\rho_2)\right] + \pi\left[\lambda B(r,\rho_2)+\lambda\rho_1{}^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right]\right] \tag{28}$$

Using $\tilde{S}$ instead of $\overline{S}$ in the objective function gives an additively separable objective function, that is:

$$\overline{TC}(r,s) = TC_1(s)+TC_2(r) \tag{29}$$

Where:

$$TC_1(s) = \frac{h_1}{1-\rho_1}\left[\frac{\rho_1(\rho_1^s-1)-s\rho_1+s}{1-\rho_1^{s+1}}\right] + \pi\lambda\rho_1{}^s\left[\frac{1-\rho_1}{1-\rho_1^{s+1}}\right] \tag{30}$$

$$TC_2(r) = h_2\left[r-\rho_2+\rho_2 B(r,\rho_2)\right] + \pi\lambda B(r,\rho_2) \tag{31}$$

Remark: We already have shown that $\overline{S}$ and $TC(r,s)$ are convex functions. Interestingly, it is easy to show that $\tilde{S}$ and $\overline{TC}(r,s)$ are also convex functions.

6.1 Optimal values of s and r

Let s^* and r^* be the optimal values of s and r, respectively; and let $\Delta TC_1(s) = TC_1(s+1)-TC_1(s)$ and $\Delta TC_2(r) = TC_2(r+1)-TC_2(r)$. As $TC_1(s)$ and $TC_2(r)$ are convex, s^* and r^* are the smallest positive integers which satisfy the conditions $\Delta TC_1(s^*)\geq 0$ and $\Delta TC_2(r^*)\geq 0$. The functions $\Delta TC_1(s)$ and $\Delta TC_2(r)$ are calculated as below:

$$\Delta TC_1(s) = \frac{\rho_1^{s+1}\left(-\pi\lambda(\rho_1-1)^2+sh_1(\rho_1-1)+h_1(\rho_1-2)\right)+h_1}{\left(-1+\rho_1^{s+2}\right)\left(-1+\rho_1^{s+1}\right)} \tag{32}$$

$$\Delta TC_2(r) = \lambda\left(\frac{h_2}{\nu}+\pi\right)\left[B(r+1,\rho_2)-B(r,\rho_2)\right]+h_2 \tag{33}$$

Remark: Similar conditions for above-mentioned optimality is presented in (Haji et al. 2011).

7. Numerical Examples

Optimal solutions s^* and r^* are obtained from (32) and (33). In some examples, as presented in table 2, we solved the set of balance equations (1) and (2) for $s = s^*$ and $r = r^*$. Then, we used the solutions to calculate the exact value of the objective function, $TC'\left(r^*, s^*\right)$. In these examples we aimed to assess the performance of the proposed approximation. C++ and Maple 14.0 have been utilized to obtain numerical examples.

Table 2: Numerical examples

	λ	μ	ν	h_1	h_2	π	r^*	s^*	$TC\left(r^*, s^*\right)$	$TC'\left(r^*, s^*\right)$	Error (%)
1	80	120	100	2	2	15	5	10	33.15	33.08	0.21
2	100	150	120	5	5	12	5	8	69.81	69.66	0.22
3	100	150	80	5	5	12	6	8	72.81	72.62	0.25
4	100	180	80	5	5	12	6	6	65.85	65.58	0.42
5	100	150	80	10	10	12	5	6	128.00	126.46	1.22
6	100	150	120	10	10	12	4	6	123.89	122.32	1.28
7	100	180	120	10	10	1	3	2	56.52	55.34	2.13
8	100	150	80	5	5	1	4	3	39.11	38.28	2.16
9	125	150	100	5	5	2	5	6	58.53	58.26	0.46
10	125	150	80	10	10	12	6	10	164.57	163.73	0.51
11	125	150	80	5	5	12	6	13	99.83	99.24	0.60
12	125	150	120	10	10	4	4	6	114.27	112.27	1.78
13	125	150	75	10	10	1	4	3	72.72	70.71	2.85

8. Conclusion

Some authors believe that the mathematical background of lost-sales analyses are more difficult compared to those of backordered (Bijvank and Vis 2011). However, we believe that it is more logical to formulate consumer purchase behavior as lost-sales (Huh et al. 2009, Bijvank and Vis 2011). Extensive attentions have been paid to lost-sales inventory policy during the past decades. This paper deals with stochastic parameters, base-stock inventory management policy and lost sales. Other studies in the area of lost-sales with variety of assumptions on parameters (lead time and restocking time), number of installations in each echelon and their characteristics are well categorized and discussed in (Bijvank and Vis 2011). The idea of focusing on this problem arose from (Haji et al. 2011) which considers a two-echelon inventory system, similar to the one we investigated in this paper, but with the assumption that in case of out of stock at retailer, demands are not lost. The authors could find the exact steady-state probabilities of the Markov model in the product form. Accordingly, the exact optimal solutions for such a system were derived. In this work, we proposed an approximation for the solutions to (1) and (2). It is worth considering that the presented steady-state probabilities applies to (1) when $\zeta(n \neq s) = 1$. In this regard, if $s \to \infty$ then our approximation would be the exact solution to balance equations as presented in (Haji et al. 2011).

References

Bijvank, M. and Vis, I. F., Lost-sales inventory theory: A review, *European Journal of Operational Research*, vol. 215, no. 1, pp. 1-13, 2011.

Haji, R., Haji, A. and Saffari, M., Queueing inventory system in a two-level supply chain with one-for-one ordering policy, *J. Ind. Syst. Eng*, vol. 5, no. 1, pp. 337-347, 2011.

Huh, W. T., Janakiraman, G., Muckstadt, J. A. and Rusmevichientong, P., Asymptotic optimality of order-up-to policies in lost sales inventory systems, *Management Science*, vol. 55, no. 3, pp. 404-420, 2009.

Jagerman, D. L., Some properties of the Erlang loss function, *Bell System Tech. J*, vol. 53, no. 3, pp. 525-551, 1974.

Jagers, A. and Van Doorn, E. A., On the continued Erlang loss function, *Operations Research Letters*, vol. 5, no. 1, pp. 43-46, 1986.

Ross, S. M., Introduction to probability models, Access Online via Elsevier, 2006.

Yüceer, Ü., Discrete convexity: convexity for functions defined on discrete spaces, *Discrete Applied Mathematics*, vol. 119, no. 3, pp. 297-304, 2002.