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Nonlinear Mixed-Effects Models for Multiple Repairable Systems

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Abstract

The multiple repairable systems may present system-to-system variability owing to the factors such as operation environments and working intensities of individual systems. It may be more reasonable to assume a heterogeneity among all the systems. In this paper, we propose a nonlinear mixed-effects model (NLMM), called power law process-nonlinear mixed-effects model (PLP-NLMM) and log linear process-nonlinear mixed-effects model (LLP-NLMM), which model the failure patterns of a multiple repairable systems having monotonic trend. The purpose of this research is to provide flexible applications of mixed-effects models to the reliability analysis of multiple repairable systems, mainly based on the nonhomogeneous Poisson process (NHPP). The proposed models are applied to the failure data of the hydraulic system of load-haul-dump (LHD) machines. Finally, the results from the PLP-NLMM and LLP-NLMM are compared with those from PLP, LLP and PLP-GLMM in terms of parameter estimation and its precision.

Keywords
Log Linear Process, Minimal Repair, Nonhomogeneous Poisson Process, Nonlinear Random-Effects Model, Power Law Process

1. Introduction
Modern systems consist of numerous parts working together, making the maintenance action for the systems more difficult. In general, systems can be classified into repairable and non-repairable systems according to feasibility of maintenance activity. Repairable system is a system that can be restored to an operating condition without replacement of the entire system after some repair activity is executed. For repairable systems, the patterns of failures collected after successive repair actions are of fundamental importance. For example, if time intervals between failures are tending to become larger, it indicates reliability improvement and conversely, if these intervals are getting smaller, it indicates reliability deterioration. To model the patterns of recurrent failure data, the stochastic point processes have been employed. Among them, the nonhomogeneous Poisson process (NHPP) has received much attention in the reliability literature (see Acher and Fiengold 1984, Rigdon and Basu 2000).

From time to time, the multiple repairable systems may present system-to-system variability owing to the factors such as operation environments and working intensities of individual systems. Therefore, it may be more reasonable to assume a heterogeneity among all the systems. To model the heterogeneity among systems, (empirical and hierarchical) Bayesian methods have been applied to multiple repairable systems due to their flexibility in accounting for parameter uncertainty and allowing the incorporation of a prior knowledge into the process under study (see, e.g., Hamada et al. 2008, Reese et al. 2011, Arab et al. 2012). The system heterogeneity is covered by prior distributions of the parameters, however, there may also exist the homogeneity between individual systems. The homogeneity can be explicitly modeled by assuming common parameters in the Bayesian model. If prior distributions are unnecessarily introduced to the common parameters, the prior information employed to the common parameters can make the parameter estimation procedure more complicated. In addition, computational intensity and the choice of proper prior distribution in Bayesian approaches hesitate reliability engineers to actively apply the methods to practical applications.

In this article, we will go over the application of mixed-effects models for multiple repairable systems. Mixed-effects model, which is also called "random-effects model", is a regression type of analysis which provides a flexible, powerful tool for analyzing repeated measurements data that arise in many areas as diverse as biology, econometrics and pharmacokinetics (Davidian and Giltinan 1995). It allows explicit modeling and analysis of between-system and within-system variations, along with a common baseline for all the systems. The mixed-effects models can be easily implemented through commercial softwares such as S-PLUS® *nlme* library and SAS® NLMIXED procedure. The purpose of this research is to provide flexible applications of mixed-effects models to the reliability analysis of multiple repairable systems, mainly based on the NHPP. In biostatistics areas, there have been a few studies on the application of mixed-effects models to Poisson process data. Lawless (1987) introduced random-effects model to mammary tumors data following Poisson process. Cooil (1991) studied Poisson process models with random effects to predict the malpractice claims filed against individual physicians. However, the application of mixed-effects model seems not to be emphasized in evaluating the reliability of multiple repairable systems. Recently, Tan et al. (2007) proposed a generalized linear mixed model for the power law process to analyze recurrent failure data from multiple repairable generator systems. To my knowledge, the application of nonlinear mixed-effects models to NHPP, however, have not been observed in the reliability literature.

The rest of this paper is organized as follows. In Section 2, we illustrated several NHPP models with monotonic failure intensities function in reliability analysis of repairable systems. In Section 3, Mixed-effects models are introduced to model recurrent failure data of multiple repairable systems. Failure data collected from a variety of repairable systems are analyzed in Section 4. We conclude in Section 5 with a discussion on future research and concluding remarks.

2. Nonhomogeneous Poisson Process Model

A class of models for recurrent failure data for repairable systems is the nonhomogeneous Poisson process (NHPP). The NHPP assumes a *minimal repair* model in which the occurrence of failures and subsequent repairs tends to have a negligible effect on overall system reliability, restoring the system performance to the exact same condition as it was just before the failure. The NHPP is defining by its nonnegative intensity function $\nu(t)$. The expected number of failures in the time interval $(0, t]$ is obtained by $\Lambda(t) = \int_0^t \nu(u)\, du$. Under an NHPP,

- $N(t+h) - N(t)$, the number of failures in an interval $[t, t+h]$, has a Poisson distribution with mean $\Lambda(t + h) - \Lambda(t)$,
- $N(a_1, b_1)$ and $N(a_2, b_2)$ are independent for non-overlapping intervals (a_1, b_1) and (a_2, b_2).

The intensity function $\nu(t)$ coincides with the rate of occurrence of failures (ROCOF) associated with the repairable system (Rigdon and Basu 2000). When the intensity function is constant, i.e., $\nu(t) \equiv \nu$, the process reduces to a homogeneous Poisson process (HPP). The NHPP has been widely used in modeling failure frequency for repairable systems because of its flexibility and mathematical tractability via intensity function $\nu(t)$ (Ascher and Feingold 1984, Krivtsov 2007).

Within the class of NHPP models, one particular and most commonly used form of the NHPP in the literature is the power law process (PLP). Crow (1974) suggested a PLP model under such "find it and fix it" conditions with the intensity function

$$\nu_1(t) = \left(\frac{\beta}{\alpha}\right)\left(\frac{t}{\alpha}\right)^{\beta-1}, \quad t > 0, \tag{1}$$

where $\beta(> 0)$ and $\alpha(> 0)$ are the shape and scale parameters, respectively. The corresponding mean cumulative number of failures over $(0, t]$ is $\Lambda_1(t) = (t/\alpha)^{\beta}$. The PLP model has been employed to model the failure patterns of a repairable system having monotonic intensity, *i.e.*, decreasing failure patterns (reliability improvement) with $\beta < 1$ or increasing failure patterns (reliability deterioration) with $\beta > 1$. When $\beta = 1$, the PLP reduces to the homogeneous Poisson process (HPP) with constant failure intensity $\nu_1(t) = 1/\alpha$, and time between failures are independently exponentially distributed. Point estimators of the PLP parameters and functions thereof, are in closed form, hence easily obtainable.

As another functional form of NHPP, a log linear process (LLP) has the following intensity function

$$\nu_2(t) = \alpha e^{\beta t}, \quad t > 0, \tag{2}$$

and corresponding mean cumulative number of failures over $(0, t]$ is $\Lambda_2(t) = (\alpha/\beta)(e^{\beta t} - 1)$, for the parameters $\alpha(> 0)$ and β. The LLP model was first proposed by Cox and Lewis (1966) to model air conditioner failures.

Similar to the PLP model, the LLP can provide the flexibility in modeling the failure patterns with monotonic intensity, *i.e.*, decreasing failure patterns with $\beta < 0$ or increasing failure patterns with $\beta > 0$. When $\beta = 0$, the LLP reduces to the HPP with constant failure intensity $\nu_2(t) = \alpha$.

3. Mixed-Effects Model for Nonhomogeneous Poisson Process

Sometimes inter-system variability is substantial enough to warrant inclusion of an individual effect in the model. Such effects are referred to as "unobserved heterogeneity" (Lawless 1987). In the formulation of mixed-effects model, the unobserved heterogeneity has been explicitly incorporated into the model under study. Because the mixed-effects models can easily handle (balanced and unbalanced) repeated-measurements data, and allow to simultaneously model both between-individual and within-individual variation, they have been widely used in medical studies (Verbeke and Lesaffre 1996, Hartford and Davidian 2000).

For analyzing the reliability of multiple repairable systems, the underlying model for each individual system may be reasonably assumed to be NHPP. Based on the NHPPs with monotonic failure intensities, we will illustrate the estimation method of their parameters for the mixed-effects models. Suppose that there are m independent systems; the system i is observed over the time interval $(0, T_i)$ and n_i failures are observed to occur, at times $t_{i1} < \cdots < t_{in_i}$. For the parameters of the NHPP, $\boldsymbol{\theta}$, the likelihood function is

$$\mathcal{L}(t_{ij}; \boldsymbol{\theta}) = \prod_{i=1}^{m} \left\{\prod_{j=1}^{n_i} \nu(t_{ij}; \boldsymbol{\theta})\right\} exp\{-\Lambda(T_i)\}. \quad (3)$$

The NHPP is flexible in that the covariate information, if exists, can be explicitly modeled via the failure intensity as $\nu(t_{ij}; \boldsymbol{\theta}, \boldsymbol{\xi}) = \nu(t_{ij}; \boldsymbol{\theta}) g(\mathbf{x}; \boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is the coefficient vector for covariate x, and $g(\cdot)$ is a certain monotonic differentiable function, *e.g.*, exp(·) or log(·). By incorporating the system-to-system variation into random effects $\boldsymbol{b}_i$, along with fixed effects identical to all the systems $\boldsymbol{\beta}$, the contribution to the likelihood function (3) having observed failures n_i at times $t_{ij} (j = 1, \dots, n_i)$ for individual system i is

$$\mathcal{L}_i(\boldsymbol{\beta}) = \int_{\boldsymbol{b}_i} \left\{\prod_{j=1}^{n_i} \nu(t_{ij} | \boldsymbol{\beta}, \boldsymbol{b}_i)\right\} exp\{-\Lambda(T_i | \boldsymbol{\beta}, \boldsymbol{b}_i)\} p(\boldsymbol{b}_i)\, d\boldsymbol{b}_i,$$

where $p(\boldsymbol{b}_i)$ is the density of the random effects vector $\boldsymbol{b}_i$. The likelihood function with parameters $\boldsymbol{\beta}$ and $\boldsymbol{b}_i$ from the sample of m systems has the form

$$\mathcal{L}(\boldsymbol{\beta}) = \prod_{i=1}^{m} \int_{\boldsymbol{b}_i} \left\{\prod_{j=1}^{n_i} \nu(t_{ij} | \boldsymbol{\beta}, \boldsymbol{b}_i)\right\} exp\{-\Lambda(T_i | \boldsymbol{\beta}, \boldsymbol{b}_i)\} p(\boldsymbol{b}_i)\, d\boldsymbol{b}_i, \quad (4)$$

and maximizing the (log) likelihood yields the maximum likelihood estimate (MLE) of $\boldsymbol{\beta}$, denoted by $\widehat{\boldsymbol{\beta}}$.

4. Numerical Application

The proposed model was applied to the failure data of the hydraulic system of load-haul-dump (LHD) machines given in Kumar and Klefsjo (1992), namely LHD 01, LHD 09, and LHD 17. Such machines belong to three different groups, representing old (LHD 01), medium old (LHD 09), and new machines (LHD 17). Kumar and Klefsjo (1992) found that such data (given Table 1) showed an increasing failure intensity function, and then analyzed them in a PLP. Attardi and Pulcini (2005) analyzed the same data by using the 2-parameter Engelhardt-Bain process (2-EBP).

Table 1: Failure times of hydraulic system in LHD machines

LHD 01	327	452	459	465	572	849	903	1235	1745	1855	1865	1874	1959	1986
	2045	2061	2069	2103	2124	2276	2434	2478	2496					
LHD 09	278	539	1529	1720	1827	1859	1910	1920	2052	2228	2475	2640	3094	3236
	3274	3523	3735	3939	4121	4237	4267	4291	4323	4361	4371	4682	4743	
LHD 17	401	437	455	614	955	1126	1150	1500	1572	1875	1909	1954	2278	2280
	2350	2407	2510	2521	2526	2529	2673	2753	2806	2890	3108	3230		

By using the NLMM under the assumptions of PLP and LLP, called PLP-NLMM and LLP-NLMM, we analyzed the failure data of LHD machines in Table 1. The estimates of fixed-effect parameters are presented in Table 2, along with the estimates of the random-effect parameters in Table 3. Based on the results of Table 2 and Table 3, the parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ of the PLP (crow 1974), LLP (Cox and Lewis 1966), PLP-GLMM (tan et al. 2007), PLP-NLMM and LLP-NLMM are listed in Table 4.

Table 2: Estimates of the fixed-effect parameters for three sets of LHD machine data

Fixed-effects parameter	**PLP-GLMM**	**PLP-NLMM**	**LLP-NLMM**
$\hat{\beta}_1$	-6.9324	411.8202	0.0036
$\hat{\beta}_2$	1.2354	1.5022	0.0004

Table 3: Estimates of the random-effect parapmeters for three sets of LHD machine data

ID	**Random-effects parameter**	**PLP-GLMM**	**PLP-NLMM**	**LLP-NLMM**
LHD 01	$\hat{b}_{11}$	0.2620	-85.5739	0.0006
	$\hat{b}_{12}$	0.0007	2.651×10^{-8}	0.0001
LHD 09	$\hat{b}_{21}$	-0.3731	132.2355	-0.0008
	$\hat{b}_{22}$	-0.0010	-4.1438×10^{-8}	-0.0002
LHD 17	$\hat{b}_{31}$	0.1111	-46.6616	0.0002
	$\hat{b}_{32}$	0.0003	1.4928×10^{-8}	3.7207×10^{-5}

Table 4: Parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ for three sets of LHD machine data

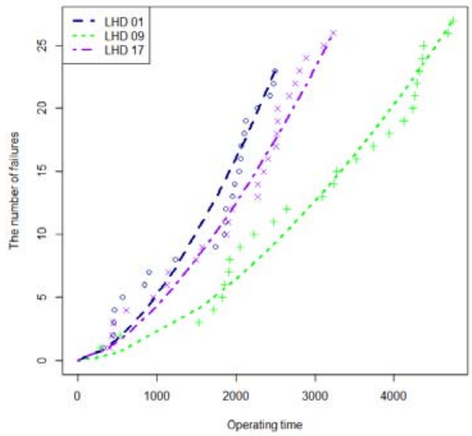
ID	**parameter**	**PLP**	**PLP-GLMM**	**PLP-NLMM**	**LLP**	**LLP-NLMM**
LHD 01	$\hat{\alpha}_1$	363.8135	220.6463	326.2463	0.0038	0.0042
	$\hat{\beta}_2$	1.6281	1.2361	1.5022	0.0006	0.0005
LHD 09	$\hat{\alpha}_1$	646.5576	371.8304	544.0557	0.0024	0.0027
	$\hat{\beta}_2$	1.6539	1.2344	1.5022	0.0003	0.0003
LHD 17	$\hat{\alpha}_1$	383.2002	249.7326	365.1586	0.0043	0.0038
	$\hat{\beta}_2$	1.5284	1.2356	1.5022	0.0003	0.0005

Using the MLEs for the five model, we can obtain the expected number of failures $\Lambda(t)$. Based on the results of Table 4, the mean absolute error (MAE) between the observed number of failures *N(t)* and the expected number of failures $\Lambda(t)$ are listed in Table 5. The LLP-NLME yields the smallest MAE for three sets of LHD machine data, while the PLP-GLMM results in the largest MAE for three sets of LHD machine data.

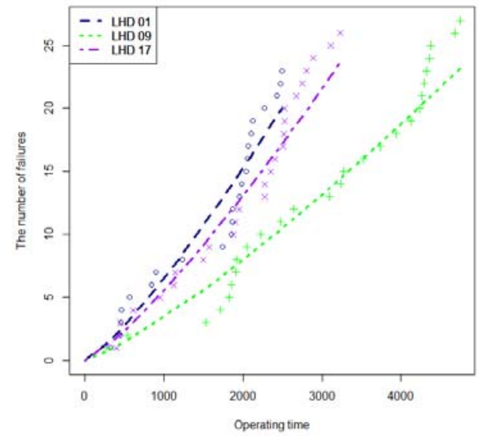
Figure 1 depicts the observed number of failures *N(t)* along with the expected number of failures $\Lambda(t)$ under the five model assumptions for three sets of LHD machine data. Figure 1 show that the PLP-NLMM and LLP-NLME provide the better representation for three sets of LHD machine data.

Table 5: MAE between between the observed number of failures *N(t)* and the expected number of failures $\Lambda(t)$ for three sets of LHD machine data

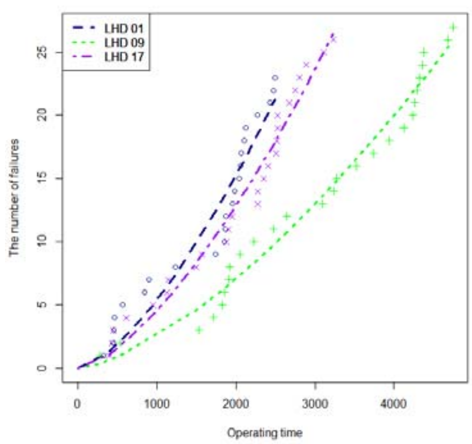
ID	**PLP**	**PLP-GLMM**	**PLP-NLMM**	**LLP**	**LLP-NLMM**
LHD 01	1.6340	1.6555	1.5778	1.3644	1.3562
LHD 09	1.1296	1.2975	1.0874	1.1225	0.9779
LHD 17	1.0499	1.3871	1.0369	1.1141	0.9429



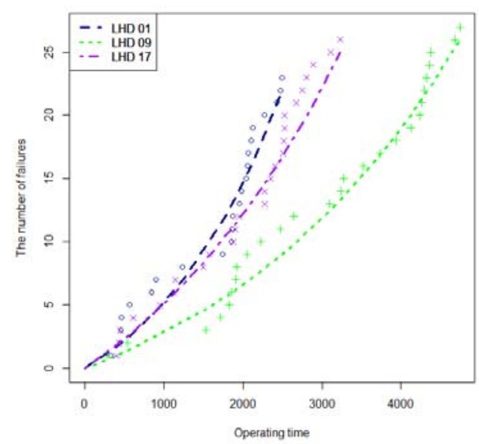
(a) PLP



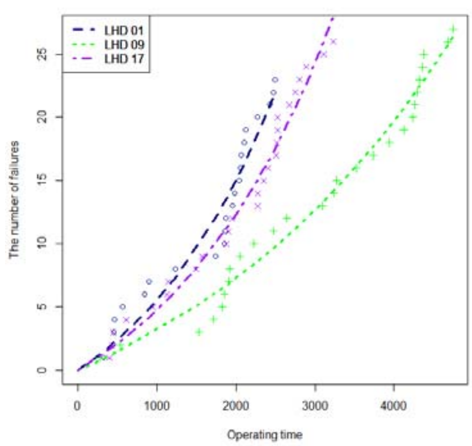
(b) PLP-GLMM



(c) PLP-NLMM



(d) LLP



(e) LLP-NLME

Figure 1. The observed number of failures $N(t)$ along with the expected number of failures $\Lambda(t)$ under the PLP, PLP-GLMM, PLP-NLMM, LLP and LLP-NLME assumption for three sets of LHD machine data

5. Conclusion

In this paper, we propose a nonlinear mixed-effects model (NLMM), called PLP-NLMM and LLP-NLMM, which model the failure patterns of multiple repairable systems having monotonic trend. The purpose of this research is to provide flexible applications of mixed-effects models to the reliability analysis of multiple repairable systems, mainly based on the NHPP. The proposed models are applied to analysis the failure data of the hydraulic system of some load-haul-dump (LHD) machines. The analysis results show that the PLP-NLMM and LLP-NLMM provide the better representation for three sets of LHD machine data.

Both the PLP and the LLP are able to model the failure pattern of a repairable system with monotonic trend. However, these models cannot be appropriately described when a non-monotonic trend(e.g. *bathtub shaped* intensity function) in the failure data is observed. A repairable system is subject both to early failures due to the presence of defective parts or assembly defects that are not screened out completely through the burn-in process, as well as wear-out failures caused by deterioration phenomena (Pulcini, 2001). This is called the *bathtub shaped* intensity function, which is typical for large and complex equipment with a number of different failure modes (Nelson (1988)). Pulcini (2001) proposed the superposed power law process (S-PLP) to model the bathtub shaped intensity function. More recently, Mun et al. (2013) proposed the superposition of two LLPs (called the superposed log-linear process) to model the bathtub shaped intensity function. Future research directions include a mixed-effects model for multiple repairable systems with bathtub shaped intensity function.

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Biography

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