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Financial and Reliability Evaluation on Multi-State Weighted *k*-out-of-*n*: F System

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Abstract

A financial view has been proposed for reliability evaluation of multi-state weighted *k*-out-of-*n*:F systems. The cost which is imposed on the components by failures is employed as reliability index. It is used to define a weight for each component. It also provides an opportunity to analyze the reliability of functioning periods in addition to design time in evaluation process. Furthermore, time value of money can be considered in system reliability assessment. The Universal Generating Function (UGF) has been applied to estimate the multi-state weighted *k*-out-of-*n*:F system reliability based on components' reliability measures. At last, the present value of failure cost is calculated using system reliability measures. An illustrative example is provided for the proposed system reliability evaluation.

Keywords
Multi-state weighted *k*-out-of-*n*:F system; Reliability evaluation; Failure cost; universal generating function (UGF).

1. Introduction
The main goal of reliability engineering is to generate reliable systems (Tavakkoli-Moghaddam et al. 2008). System reliability evaluation (prediction) is a preliminary step to reach the reliable systems (Saleh and Marais 2006). System evaluation usually precedes system improvement. System structure, system state, and reliability index of the components are used for system reliability estimation. System reliability optimization models provide an opportunity to consider system cost in creating reliable systems (Khorshidi et al. 2013). As a result, combination of cost in evaluating of the system reliability has a significant importance.

The *k*-out-of-*n* systems are popular system structures with a wide application in real systems. The multi-engine aircraft, the multi-display System Cockpit, the multi-transmitter system in communication system, and the oil supply system with multiple pipelines are some examples to make illustration about *k*-out-of-*n* systems (Kuo and Zou 2003; Tian et al. 2009). A *k*-out-of-*n*:F(G) system fails (works) if and only if at least k components fail (work). A *1*-out-of-*n*:F system is a series system, and a *n*-out-of-*n* system is a parallel system (Yamamoto et al. 2011). Also, a weight which shows the performance of each component can be allocated. Therefore, *k* would be the summation of allocated weights for the mentioned components (Li and Zuo 2008). Generally, a weighted *k*-out-of-*n* system can cover the whole features of the *k*-out-of-*n* systems (Wu and Chen 1994a). Industrial systems are almost multi-state systems in which components have different weights in different states. Consequently, a multi-state weighted *k*-out-of-*n*:F system is in state *j* or below if the total weight of all components is equal to or greater than a threshold value k_j.

The reliability of consecutive weighted *k*-out-of-*n*:F systems has been computed with minimal cut sets recursively in Wu and Chen (1994b). Serkan and Yazgi Tutuncu (2009) proposed a recursive formula to evaluate the reliability of consecutive weighted *k*-out-of-*n*:F systems with binary components. Yamamoto et al. (2011) use a recursive algorithm to estimate multi-state *k*-out-of-*n*:F systems. In Ding et al. (2012), a definition for multi-state weighted *k*-out-of-*n*:F systems is introduced, and the system is evaluated by UGF method. However, they defined the weight the same as k-out-of-n:G systems.

Cost is an index that managers can simply understand (Rhee and Ishii 2003), so it is employed for reliability evaluation in this paper. An available component can generate income for the system (Marais and Saleh 2009). Hamadani and Khorshidi (2013) use the income generated by the components to evaluate the series-parallel systems and develop an optimization model. As a result, failure or being in deteriorated state leads to losing the income. Therefore, it can be mentioned as a failure cost. This cost in different reliability levels (states) is used as a reliability index for reliability evaluation of k-out-of-n:F systems. This index can provide a valuable opportunity to consider time dimension in reliability evaluation. Subsequently, working periods of a k-out-of-n:F system would be included in evaluation procedure. Also, time value of money can be employed to compute the present value of system reliability.

In general, this paper proposes a model for computing multi-state weighted k-out-of-n:F system reliability based on failure cost by universal generating function (UGF) approach. Failure cost is considered as weight. Since different components can generate different level of income in their different states (Hamadani and Khorshidi 2013), there are different levels of failure cost for the system (multi-state weighted k-out-of-n:F system). The components' state distribution is calculated via the Markov chain. Also, a numerical example is provided to illustrate the evaluation method.

2. System reliability evaluation

According to the abovementioned statements, the functioning periods of k-out-of-n:F system in addition to design time have been studied. It can provide decision making for the system during functioning period. Also, the income which is missed by each component through time periods is employed to show the weight of the failure. Therefore, this index can provide using engineering economics' techniques to calculate the present value of the system reliability. The components are multi-state which have different income generating rate in different states. Firstly to evaluate this system, the probability of being the components in each state in different time periods should be determined. The considered assumptions are as below:

- The time to failure of each component is independent.
- Each component starts working in perfect functioning state.
- Since the components in k-out-of-n:F systems work parallel, the failure cost of the system is equal to the summation of components' cost.

2.1. Determining the probability

Each component through each time period can transmit from each state to another state or can remain in the same state with an identified probability. The transition probability from state *i* to state *j* is denoted by p_{ij}. Each component may change its state to lower or higher state by failures or maintenance respectively. Transition probabilities are shown in a transition matrix (TM) as Eq. 1.

$$\text{TM}=\begin{bmatrix} p_{00} & \cdots & p_{0M} \\ \vdots & \ddots & \vdots \\ p_{M0} & \cdots & p_{MM} \end{bmatrix}, \quad \text{and} \quad \forall i,\ \sum_{j=0}^{M} p_{ij} = 1 \tag{1}$$

Figure 1 is provided to illustrate the components' states and the state transitions during the time periods. In this diagram, each component has three different states $\{0,1,2\}$ that 0 is complete failure and 2 is perfect functioning. As it is assumed, the component at the starting point is in state 2. The component works in four periods, and can remain in its previous state or change to other states during each period.

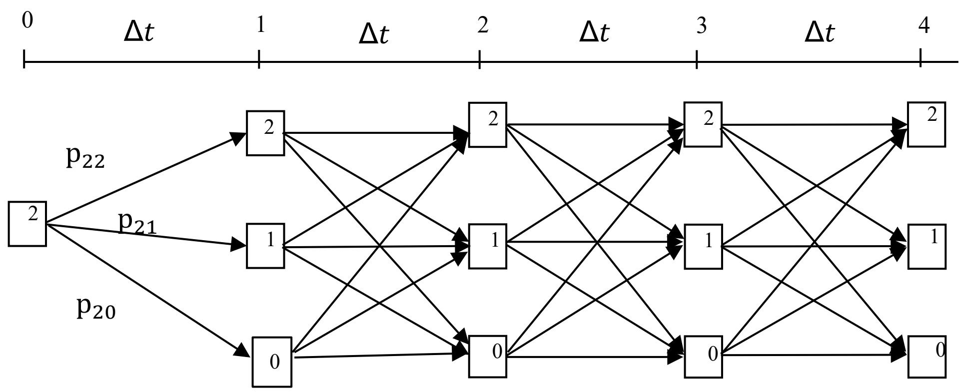

Figure 1. Different states of a component during the operational periods

Based on the transition probabilities, the probability of being component *i* in state *j* at time *t* ($P_j^i(t)$) is as expressed in Eq. 2.

$$P_j^i(t) = \sum_{k=0}^{M} P_k^i(t-1)p_{kj}\,, \qquad i = 1, \dots, n;\ t = 1, \dots, T \tag{2}$$

where at design time (t=0): $P_M^i(0) = 1,\ P_0^i(0) = \cdots = P_{M-1}^i(0) = 0$

The system is structured as weighted *k*-out-of-*n*:F system. The UGF computes the probability of being the whole system in state *j* at time *t*. After that, the present value of whole system could be calculated based on the system probability distribution, their equivalent failure cost, and interest rate.

2.2. System evaluation by UGF

The computed probabilities, by Eq. 2, have been employed to create the u-function of component *i* in state *j* at time *t* ($u_t^i(z)$) as Eq. 3 (for more information about UGF see Levitin (2005)):

$$u_t^i(z) = P_0^i(t)z^{Fc_0^i} + \cdots + P_M^i(t)z^{Fc_M^i} = \sum_{j=0}^{M} P_j^i(t)z^{Fc_j^i} \tag{3}$$

where Fc_j^i is failure cost of component *i* in state *j*.

Based on the u-function of the components, the u-function for the system at time *t* ($U_t(z)$) can be calculated with operator $\otimes_+$ via Eqs. 4 and 5:

$$\otimes_+ \left(\sum_{j_i=0}^{k_i} p_{ij_i} z^{x_{ij_i}}\right) = \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \dots \sum_{j_n=0}^{k_n} \left(\prod_{i=0}^{n} p_{ij_i} z^{\sum_{i=1}^{n} x_{ij_i}}\right) \tag{4}$$

$$U_t(z) = \otimes_+ \left(u_t^1(z), u_t^2(z), \dots, u_t^n(z)\right) = \sum P z^{Fc} \tag{5}$$

where Fc is the failure cost of the system.

$R_t^s(k_j, n)$ is the probability for the system to be in state *j* (has failure cost k_j) at time *t*. This parameter is computed as below by Eq. 6:

$$R_t^s\left(k_j, n\right) = \sum P\,, \quad \text{for} \quad k_j \le Fc < k_{j+1} \tag{6}$$

The present value for the system's failure cost would be calculated according to Eq. 7:

$$PV_F = \sum_{j=0}^{M} \sum_{t=0}^{T} R_t^s\left(k_j, n\right).\, k_j/(1+r)^t \tag{7}$$

where *r* is interest rate. It can provide an opportunity to consider other system costs. As a result, system reliability and system cost can be compared together simply.

3. Numerical example

To illustrate the proposed reliability evaluation of k-out-of-n:F system, a numerical example is introduced. Consider a system with three different components. Each component has three possible states. Also, there are four functioning time periods (Figure 1). Each component can generate different level of income in different states according to Table 1.

Table 1. Income generating distribution of components

Component | state	0	1	2
1	0	2	3
2	0	3	4
3	0	3	5

Accordingly, each component can have equivalent costs which are caused by failures in different states that is presented in Table 2.

Table 2. Failure cost distribution of components

Component | state	0	1	2
1	3	1	0
2	4	1	0
3	5	2	0

For the whole system, the system is in state 0 if the total failure cost is more than 10 ($k_0 = 10$); the failure cost in [4,10) interval places the system in state 1 ($k_1 = 4$); if the total failure cost is between 0 and 4, then the system is in state 2 ($k_2 = 0$).

Table 3 shows the transition probabilities for component 1 in different states.

Table 3. TM for component 1

State | state	0	1	2
0	0.5	0.35	0.15
1	0.2	0.7	0.1
2	0.05	0.15	0.8

Table 4 shows the transition probabilities for component 2 in different states.

Table 4. TM for component 2

State | state	0	1	2
0	0.45	0.3	0.25
1	0.15	0.65	0.2
2	0.03	0.07	0.9

Table 5 shows the transition probabilities for component 3 in different states.

Table 5. TM for component 3

State | state	0	1	2
0	0.5	0.4	0.1
1	0.25	0.7	0.05
2	0.05	0.25	0.7

Besides, the interest rate is considered as 0.1 (r=0.1).

First of all, the probability of being each component in different states through different periods should be computed by Eq. 2 using transition matrices.

$$\begin{cases} P_2^1(1) = 0 \times 0.15 + 0 \times 0.1 + 1 \times 0.8 = 0.8 \\ P_1^1(1) = 0 \times 0.35 + 0 \times 0.7 + 1 \times 0.15 = 0.15, \\ P_0^1(1) = 0 \times 0.5 + 0 \times 0.2 + 1 \times 0.05 = 0.05 \end{cases} \qquad \sum_{j=0}^{2} P_j^1(1) = 1$$

According to this way all probabilities could be calculated. Table 6 shows the calculated probability of being in each state in different periods for component 1.

Table 6. Probability distribution of component 1

State | period	0	1	2	3
0	0	0.05	0.095	0.129
1	0	0.15	0.243	0.302
2	1	0.8	0.662	0.569

Table 7 shows the calculated probability of being in each state in different periods for component 2.

Table 7. Probability distribution of component 2

State | period	0	1	2	3
0	0	0.03	0.051	0.066
1	0	0.07	0.117	0.15
2	1	0.9	0.832	0.784

Table 8 shows the calculated probability of being in each state in different periods for component 3.

Table 8. Probability distribution of component 3

State | period	0	1	2	3
0	0	0.05	0.123	0.179
1	0	0.25	0.37	0.435
2	1	0.7	0.507	0.386

Subsequently, the UGF approach is employed to estimate the system reliability. The u-functions for components are obtained by Eq. 3. Based on the components' u-functions, the u-function of the system and system probability at different times can be calculated by Eqs. 5 and 6. Table 9 shows the calculated probability distribution of the whole system by UGF for different states and time periods.

Table 9. The probability distribution of the system using UGF

State | period	0	1	2	3
0	0	0.001	0.002	0.005
1	0	0.095	0.215	0.308
2	1	0.904	0.783	0.687

Therefore the probability for the system at time 1 to be in state 2 is 0.904, in state 1 is 0.095, and in state 0 is 0.001. As it is expected, the probability of being in state 2 is decreasing, and the probability of being in state 0 is increasing during time periods. With using these probabilities, the expected value of the system failure in each period can be computed. The cash flow of the value is shown in Figure 2.

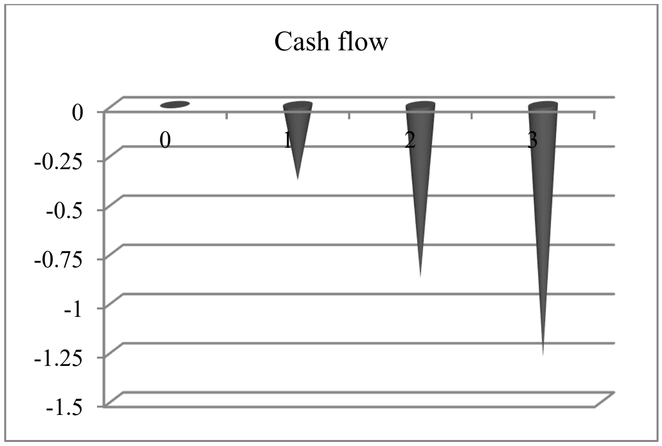

Figure 2. The failure cost cash flow

Based on the cash flow, the present value of the failure cost for the system is calculated using interest rate as below.

$$PV_F = \sum_{j=0}^{2}\sum_{t=0}^{3} R_t^s(k_j,3).k_j \Big/ (1+0.1)^t = 2.04$$

This value can help managers to decide about improving strategies. If the strategies reduce the system's present value more than their cost, they can be considered for implementation. Also, this value provides a measure for ranking the strategies.

Since this value is generated by failure cost, it denotes the present value of the failure. As a result, it should be transformed to a positive value to represent the present value of system's reliability. Since there is a relationship between the failure probability (unreliability) and reliability (Xie et al. 2004) as Eq. 8, the transformation can be used similarly.

$$R(t) = 1 - F(t) \tag{8}$$

To reach this goal, the present value of the system with no failure (perfect income) is considered. In other word, given all components are in state 2 for all time periods. Therefore for this case, the system can have 10 income generated for all periods. The present value of the mentioned system is computed as below:

$$PV_{Perfect} = \sum_{t=0}^{3} \frac{10}{(1+0.1)^t} = 34.87$$

Figure 3 shows the cash flow of two situations together.

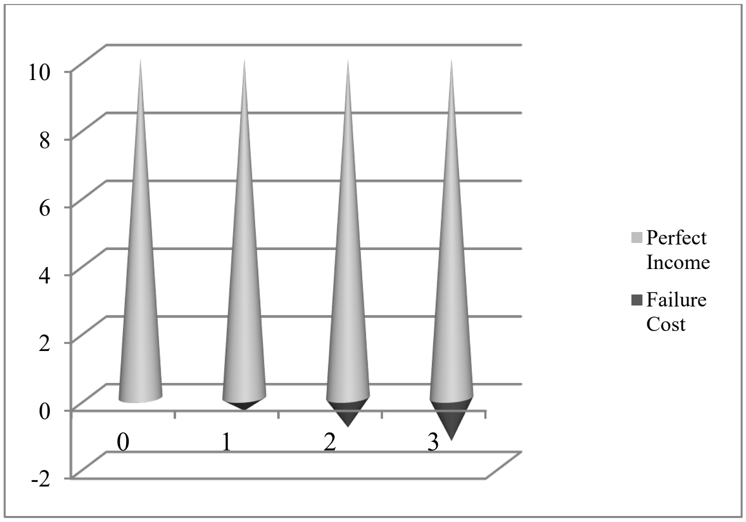

Figure 3. final cash flow

As a result, the difference between these two amounts is the present value of the system reliability (Eq. 9).

$$PV_R = PV_{Perfect} - PV_F \quad (9)$$

Consequently, the present value of system reliability for the *k*-out-of-*n*:F system is computed by Eq. 9. The PV_R, which comes from the difference of perfect situation and failure cost as 34.87-2.04, is equal to 32.83.

4. Conclusion

In this paper, the reliability of multi-state weighted *k*-out-of-*n*:F systems has been evaluated using the components' failure cost. The failure cost is employed as component weights. It facilitate either evaluating *k*-out-of-*n*:F systems or using financial techniques. Also, using failure cost as reliability index can provide an opportunity to investigate other system costs such as installation cost, maintenance, and replacement into system reliability evaluation. The system reliability is estimated by the UGF approach. Finally, the system present value is calculated based on system failure cost and system probabilities. Furthermore, the present value of system reliability is obtained using the failure cost's present value. At last, an example is used to illustrate the evaluation method.

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Biography

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