

Minimizing Makespan in Simple Assembly Lines: A MILP Approach Considering Learning and Fatigue Effects

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Abstract

This study presents a mixed-integer linear programming (MILP) model designed to minimize the makespan of simple assembly lines by integrating the effects of learning and fatigue among workers. A learning curve model with a fatigue-dependent learning rate is incorporated to capture variations in worker efficiency over time. The proposed model optimally assigns tasks and workers to stations, ensuring that individual learning abilities and fatigue accumulation are considered in the optimization process. To validate the approach, the model is applied to a benchmark problem, where different worker assignment strategies are analyzed. The results indicate that random worker allocation and balanced workload distribution can lead to significantly longer makespans, emphasizing the critical role of aligning task assignments with workers' learning capacities and fatigue progression. By accounting for these dynamic factors, the proposed model enhances the realism and effectiveness of assembly line balancing. The findings underscore the potential of integrating learning and fatigue considerations to improve efficiency in assembly line operations.

Keywords

Assembly line Balancing, Learning, Fatigue, Assembly lines.

1. Introduction

Assembly lines are a widely adopted production strategy in manufacturing industries due to their ability to efficiently organize sequential work tasks. In such systems, tasks are distributed across workstations subject to precedence and operational constraints. Two key performance metrics in evaluating the effectiveness of assembly lines are cycle time—the longest time required at any single station—and makespan, defined as the time required to complete a predetermined production batch. Task assignment strategies that optimize these two metrics are central to the broader challenge of assembly line balancing.

Traditional assembly line balancing (ALB) models generally assume that task times are constant and unaffected by human or contextual variability. Under these assumptions, evenly distributing workload across stations is considered optimal for minimizing makespan. However, in many real-world manufacturing environments—especially those involving short- and medium-run production—task times can change over time due to worker learning and fatigue. These human factors introduce variability into task execution times, which can significantly influence the overall efficiency of the production process.

Learning effects, typically represented by learning curves, describe how workers become faster at performing tasks with repeated practice. These effects are especially pronounced at the beginning of production runs, where improvements in speed are substantial. On the other hand, fatigue—a progressive decline in physical or cognitive capacity—can slow performance and counteract the gains from learning. While each of these effects has been studied

independently, few analytical models have simultaneously addressed their combined impact on task execution in assembly lines.

1.1 Objectives

This paper presents a Mixed-Integer Linear Programming (MILP) model developed to minimize the makespan of simple assembly lines by incorporating both learning and fatigue effects. The model introduces a fatigue-dependent learning rate, capturing how task times evolve with repetition and how fatigue alters the pace of learning. It optimally assigns both tasks and workers to stations while accounting for individual variability in performance dynamics. The main contributions of this study are:

- The formulation of a novel MILP model that explicitly integrates learning and fatigue to minimize the makespan in assembly line operations.
- An evaluation of the model on a benchmark problem, comparing its performance with traditional task assignment strategies such as random worker allocation and balanced workload distribution. The results show that failing to consider learning and fatigue can lead to significantly longer makespans.

The remainder of this paper is organized as follows: Section 2 reviews the relevant literature and introduces the learning-fatigue curve used in this study. Section 3 presents the MILP model. Section 4 describes the computational experiments and discusses the results. Section 5 concludes the paper and outlines directions for future research.

2. Literature review

Minimizing makespan in assembly lines with human workers requires accounting for task time variability, particularly due to the learning effect—especially in small to medium batch sizes. Failing to consider this performance evolution may lead to increased idle time, bottlenecks, and a longer makespan. (Globerson and Shtub 1984; El Abidine et al. 2024). Cohen et al. (2006) demonstrated that under homogeneous learning conditions, optimal makespan is achieved through unbalanced task distributions. In later work, Cohen et al. (2008) extended their study to heterogeneous learning environments, confirming that tailoring task allocation to worker learning rates improves performance. Bukchin and Wexler (2016) incorporated deterministic, heterogeneous learning into a linear programming model and found that sequencing slower learners first reduces makespan effectively. Koltai and Kalló (2017) investigated how makespan responds to changes in learning rates, highlighting that accurate learning rate estimates are crucial, especially for small batch sizes, due to their significant effect on makespan. Ranasinghe et al. (2024) developed a dynamic model that integrates stochastic processing times and heterogeneous learning rates, showing that ignoring learning-related variability leads to systematic underestimation of makespan.

Despite the progress made, most existing studies treat learning in isolation and do not explicitly address performance degradation over time due to fatigue. As a result, the joint impact of learning (which reduces task times) and fatigue (which may increase them) on makespan remains largely unexplored. To address this gap, our work introduces a new optimization model that simultaneously accounts for both effects, enabling more accurate and human-centered scheduling in variable performance environments.

2.1 Learning curve models

In this research, we adopt the Learning-Fatigue Curve (LFC) developed by Asadayoobi et al. (2021) as a performance modeling tool to capture the combined effects of learning and fatigue during repetitive tasks. The LFC is an extension of the classical Wright Learning Curve (WLC) introduced by Wright (1936), which models performance improvement as a power function of cumulative output under the assumption of a constant learning rate. However, the WLC does not account for fatigue, which can significantly influence worker performance over time (Benson 1968; Carron and Ferchuk 1971). The LFC (See Figure 1) addresses this limitation by introducing a fatigue-dependent learning exponent that varies with the number of repetitions. The model is defined as:

$$T_n = T_1 n^{-l(n)} \quad (1)$$

Where:
$$l(n) = l + f_p + 1 - \exp(f_p(n^{f_p} - 1)) \quad (2)$$

- T_n : time to perform the n_{th} repetition
- T_1 : time to perform the first repetition

- n : cumulative number of repetitions
- l : base learning exponent, typically in the range $[0.15, 0.52]$, corresponding to a learning rate (L) between 0.7 and 0.9.
- F_p : fatigue accumulation index, typically in the range $[0.15, 0.25]$, where higher values represent faster fatigue buildup.
- $l(n)$: learning exponent as a function of n , capturing both learning and fatigue effects.

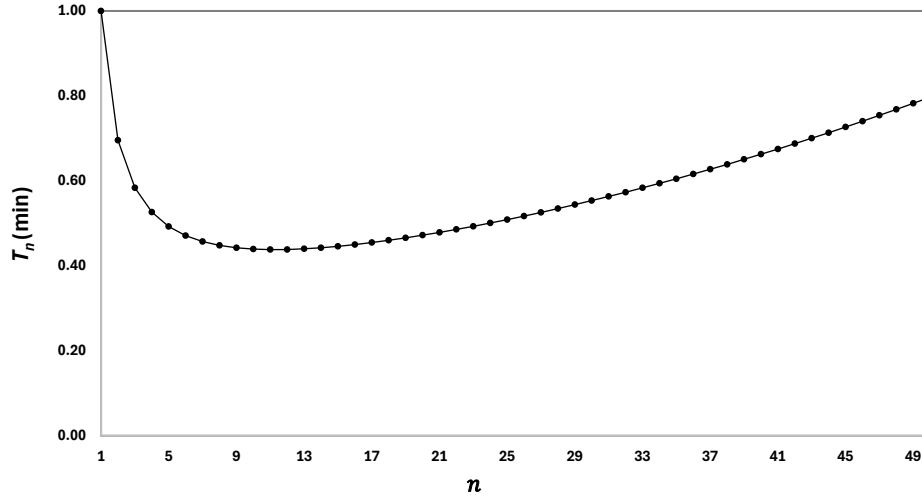


Figure 1. Illustration of the Learning-Fatigue Curve for $T_1=1$ min, $l=0.322$ ($L=0.8$), and $F_p=0.25$

3. Model description

This section describes a MILP model that assigns tasks and workers to workstations in an assembly line with the objective of minimizing the makespan, considering that task processing times vary due to the combined effects of learning and fatigue.

The time required to perform a task depends on the number of repetitions. Let t_{in} be the time to perform task i at its n_{th} repetition. Task durations are updated using a learning-fatigue curve, defined as follows:

$$t_{in} = t_{i1} n^{-l(n)} \quad (3)$$

Where:

$$l(n) = l + f_p + 1 - \exp(f_p(n^{f_p} - 1)) \quad (4)$$

Tasks are denoted by the index i , with each task assigned a distinct identifier. Workstations are indexed by j , ranging from 1 to J . The total number of workers is denoted by W , and it is assumed that $W=J$, meaning each workstation is operated by exactly one worker. Workers are indexed by w .

The model includes two binary decision variables:

- $x_{ijw}=1$, if task i and worker w are both assigned to workstation j ; 0 otherwise.
- $y_{jw}=1$, if worker w is assigned to workstation j ; 0 otherwise (Table 1- Table 2).

Table 1. MILP model for minimizing the makespan with learning and fatigue effects

$Min(M)$		(5)
$\sum_{j=1}^J \sum_{k=1}^K x_{ijw} = 1$	$\forall i$	(6)
$\sum_{j=1}^J j(x_{ijw} - x_{kjw}) \geq 0$	$\forall (i, k) k \in P_i$	(7)
$x_{ijw} \leq y_{jw}$	$\forall i, j, w$	(8)
$\sum_{j=1}^J y_{jw} \leq 1$	$\forall w$	(9)
$\sum_{w=1}^W y_{jw} \leq 1$	$\forall j$	(10)
$u_s \geq \sum_i x_{ijw} * t_{i,(s-j+1)}$	$\forall (j, s) s < J, s - N + 1 \leq j \leq s$	(11)
$v_n \geq \sum_i x_{ijw} * t_{i,(n-j+J)}$	$\forall (j, n) 1 \leq n + J - j \leq N$	(12)
$M = \sum_n v_n + \sum_{s < J} u_s$		(13)

The objective of the model is to minimize the total production time, referred to as the makespan (M), required to complete a batch of N parts. The optimization model consists of the objective function (Equation 5) and a set of constraints (6)–(13), described as follows:

- **Constraint (6)** ensures that each task is assigned to exactly one worker at one workstation.
- **Constraint (7)** enforces the precedence relations among tasks.
- **Constraint (8)** ensures that when a task is assigned to a workstation, a worker must also be assigned to that workstation.
- **Constraint (9)** restricts each worker to at most one workstation.
- **Constraint (10)** ensures that each workstation is operated by at most one worker.
- **Constraint (11)** defines the cycle times during the initial phase of production (run-up phase), in which no parts have yet been completed and the first part progresses through stations $s < J$.
- **Constraint (12)** ensures that the cycle time for completing the n_{th} part is not shorter than the processing time of the bottleneck workstation, considering learning- fatigue adjusted task durations.
- **Constraint (13)** defines the makespan as the sum of cycle times across all $J+N-I$ production cycles.

The production of a batch of size N is completed over $J + N - I$ cycles. In the first $J - I$ cycles (run-up phase), the initial part enters stations 1 through $J - I$ sequentially. From cycle J onward, one part is completed per cycle until the N_{th} part is produced. In the final $J - I$ cycles (run-out phase), the remaining parts exit the system as the occupied

stations become idle. The duration of each cycle is determined by the workstation with the maximum processing time, which varies depending on the learning-fatigue effect.

Table 2. Summary of notations applied in the model

Summary of notations:	
Indices:	
i, k	- index of tasks ($i=1\dots, I; k=1\dots, I$),
j, s	- index of workstations ($j=1,\dots, J; s=1,\dots, J$),
w	- index of workers ($k=1,\dots, W$),
n	- index of parts produced ($n=1,\dots, N$).
Parameters:	
l	- Wright's learning exponent, which ignores fatigue ($l=\log(L)/\log(2)$),
$l(n)$	- learning exponent as a function of the number of repetitions,
f_p	- physical fatigue index, specifying physical fatigue accumulation rate,
I	- number of tasks,
J	- number of workstations,
N	- number of parts produced,
t_{in}	- time necessary to execute task i in the n_{th} repetition ($t_{in} = t_{i1}n^{-l(n)}$),
W	- number of available workers ($W = J$).
Sets:	
P_i	- set of indices of those tasks which must be finished before task i is started.
Decision variables:	
x_{ijw}	- 0-1 decision variable, if $x_{ijw}=1$, then task i is performed by worker w at workstation j , otherwise $x_{ijw}=0$
y_{jw}	- 0-1 decision variable; if $y_{jw}=1$, then worker w is assigned to workstation j , otherwise $y_{jw}=0$,
u_s	- cycle time when executing the first part on station s , where $s < J$,
v_n	- cycle time when completing part n ,
M	- makespan or total production time.

4. Illustration of the results

To illustrate the performance of the proposed model, let us consider a benchmark problem named as *Buxey* (Scholl 1993). The precedence graph of tasks and the task times (in minutes) of the problem are given in Figure 2.

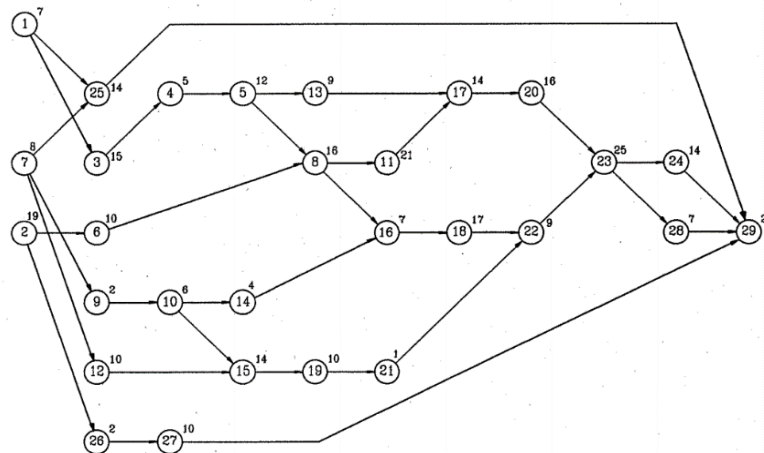


Figure 2. Precedence graph and task times of the sample problem.

The model presented in Table 1 is applied to determine the optimal makespan of the sample problem, taking into account workers' learning and fatigue effects. The AIMMS Prescriptive Analytics Platform is used to implement the algorithm and generate the results. The MILP problem is solved using CPLEX version 12.7.1.

4.1 Effect of learning and heterogenous fatigue on the makespan, tasks and workers allocations

The combined impact of worker learning and fatigue on the optimal makespan is examined using the learning-fatigue curve. To illustrate the analysis, an assembly line comprising three stations ($J=3$) and a production quantity of 50 units ($N=50$) is considered. While this example serves to demonstrate the approach, the proposed model is applicable to any assembly line configuration and production volume.

Three distinct scenarios are analyzed, each characterized by different worker learning and fatigue rates:

- **Case 1:** All three workers possess a fast learning rate ($L = 0.7$), while their fatigue accumulation rates vary from high to low: $F_p = 0.15, 0.20$, and 0.25 .
- **Case 2:** All three workers have a moderate learning rate ($L=0.8$), with the same variation in fatigue accumulation rates: $F_p = 0.15, 0.20$, and 0.25 .
- **Case 3:** All three workers exhibit a slow learning rate ($L = 0.9$), again with fatigue accumulation rates of $F_p = 0.15, 0.20$, and 0.25 .

Table 3 presents the optimal makespan, task-to-station assignments, and worker-to-station allocations for the three different scenarios. A consistent pattern emerges across all scenarios: workload is assigned to workers based on their fatigue accumulation rates. Workers with lower fatigue rates (F_p) are consistently assigned higher workloads, while those with higher fatigue rates are given lower workloads. In the fast ($L = 0.7$) and moderate ($L = 0.8$) learning cases, the workload decreases progressively across the stations (137, 120, and 67 minutes), and the worker allocation follows the order of increasing fatigue rates. These configurations result in relatively low makespans of 1642.0 and 2683.8 minutes, respectively. In the slow learning case ($L = 0.9$), the workload increases along the stations (65, 135, and 124 minutes), yet the same principle is applied: workers are assigned based on their fatigue accumulation, with the most fatigue-resistant worker handling the highest workload. This case yields a significantly higher makespan of 4348.2 minutes, primarily due to the limited effect of learning. These findings highlight the critical role of fatigue-aware workload assignment in optimizing assembly line performance. Incorporating fatigue and learning characteristics into worker allocation leads to more efficient task distribution, ultimately contributing to reduced makespan.

Table 3. Optimal worker and task assignments, initial station times, and makespan in case of learning with heterogeneous fatigue

		Station 1	Station 2	Station 3	Makespan (min)
L=0.7	Worker allocation (F_p)	0.15	0.2	0.25	1642.0
	Task allocation	1,2,3,4,5,6,7,8,9,10,12,13,14,15	11,16,17,18,19,20,21,22,23	24,25,26,27,28,29	
	Initial station times (min)	137	120	67	
L=0.8	Worker allocation (F_p)	0.15	0.2	0.25	2683.8
	Task allocation	1,2,3,4,5,6,7,8,9,10,12,13,14,15	11,16,17,18,19,20,21,22,23	24,25,26,27,28,29	
	Initial station times (min)	137	120	67	
L=0.9	Worker allocation (F_p)	0.25	0.15	0.2	4348.2
	Task allocation	1,3,4,5,7,9,10,12	2,6,8,11,13,14,16,17,18,20,26	15,19,21,22,23,24,25,27,28,29	
	Initial station times (min)	65	135	124	

4.2 Minimizing the makespan when integrating a fast-fatigued worker in the assembly line

To further examine the influence of worker learning and fatigue rates on assembly line performance, we consider a scenario in which a fast-fatigued worker (e.g., an older worker) with a fatigue rate of $F_p=0.25$ replaces a slow-fatigued worker (e.g., a younger worker) with a fatigue rate of $F_p=0.15$. The assembly line consists of three stations ($J=3$), producing a batch of 50 units ($N=50$). All workers are assumed to have a moderate learning rate ($L=0.8$). In this modified setup, two workers have a low fatigue rate ($F_p=0.15$), while one worker has a higher fatigue rate ($F_p=0.25$). The objective is to evaluate the impact of introducing the fast-fatigued worker and to determine their optimal station assignment in order to minimize the makespan.

Table 4 presents the results of the two scenarios under consideration. In Scenario 1, where all three workers have the same fatigue rate ($F_p = 0.15$), the workload is distributed in a relatively balanced manner across the stations (111, 109, and 104 minutes), resulting in a makespan of 2143.1 minutes. In Scenario 2, the introduction of one fast-fatigued worker ($F_p = 0.25$) requires a more uneven workload distribution (136, 130, and 58 minutes), with the most fatigued worker assigned to the station with the lowest workload. Despite this compensation, the makespan increases significantly to 2583.4 minutes.

The comparison between these two scenarios highlights the sensitivity of assembly line performance to worker fatigue variation. While Scenario 1 benefits from a uniform fatigue profile that enables a more balanced workload assignment, Scenario 2 shows that introducing a more fatigue-prone worker disrupts this balance and leads to lower efficiency, even when tasks are optimally reassigned. These results underscore the importance of considering fatigue characteristics in workforce planning and suggest that minimizing variability in fatigue resistance among workers contributes to improved overall line performance.

Table 4. Comparison between optimal task assignments, initial station times, and makespans for the two scenarios

		Station 1	Station 2	Station 3	Makespan (min)
Scenario 1	Worker allocation (F_p)	0.15	0.15	0.15	2143.1
	Task allocation	1,2,3,4,5,6,7,8,12,13	9,10,11,14,15,16,17,18,19,25	20,21,22,23,24,26,27,28,29	
	Initial station times (min)	111	109	104	
Scenario 2	Worker allocation (F_p)	0.15	0.15	0.25	2583.4
	Task allocation	1,2,3,4,5,6,7,8,11,13,17	9,10,12,14,15,16,18,19,20,21,22,23,26,28	24,25,27,29	
	Initial station times (min)	136	130	58	

4.3 Impact of ignoring learning and fatigue effects and randomly assigning the fast-fatigued worker in the assembly line

In this section, we investigate the impact of neglecting the learning and fatigue effects when integrating a fast-fatigued worker into the assembly line. This involves solving the model under the assumption that all workers have a constant performance over time—that is, a learning rate of $L=1$ and a fatigue rate of $F_p=0$. In reality, the system includes three workers with a learning rate of $L=0.8$: two workers have a fatigue rate of $F_p=0.15$, and one fast-fatigued worker has a fatigue rate of $F_p=0.25$.

To assess the implications of using these incorrect assumptions, we compare the makespan obtained under the simplified setting to the optimal makespan computed using the actual learning and fatigue characteristics. Both absolute and relative differences in makespan are calculated across all possible worker-to-station configurations to capture the full impact of worker assignment. The differences are computed as follows:

$$\Delta \text{Makespan} = \text{Non-optimal Makespan} - \text{Optimal Makespan} \quad (14)$$

$$\% \Delta \text{Makespan} = [(\text{Non-optimal Makespan} - \text{Optimal Makespan}) / \text{Optimal Makespan}] \times 100 \quad (15)$$

Table 5. Optimal and non-optimal worker assignments and makespans, along with absolute and relative makespan differences

	Station 1	Station 2	Station 3	Makespan (min)	Δ Makespan (min)	Δ Makespan (%)
Optimal worker allocation	0.15	0.15	0.25	2583.4	-	-
Worker allocation Scenario 1	0.15	0.15	0.25	3312.4	729.0	28.2%
Worker allocation Scenario 2	0.15	0.25	0.15	3300.4	717.0	27.8%
Worker allocation Scenario 3	0.25	0.15	0.15	3280.2	696.8	27.0%

Table 5 compares the makespan and performance deviations across different worker allocation scenarios, highlighting the impact of ignoring learning and fatigue effects. For each scenario, both the absolute difference in makespan and the percentage increase relative to the optimal makespan are reported. The results show a clear and consistent deterioration in performance when worker learning and fatigue are disregarded. Makespan increases range from 27.0%

to 28.2%, with absolute increases of up to 729.0 minutes compared to the optimal configuration. Scenario 1, which replicates the optimal allocation without accounting for learning and fatigue ($F_p = 0.15, 0.15, 0.25$), results in the highest performance loss. Even alternative worker placements, such as assigning the fast-fatigued worker to Station 1 or 2, do not yield significant improvements.

These findings underscore the critical role of accurately modeling worker learning and fatigue in assembly line balancing. Neglecting these human factors leads to consistently suboptimal outcomes, regardless of worker arrangement. This reinforces the need for fatigue- and learning-aware planning, especially when integrating variability in worker capabilities, such as the presence of a fast-fatigued worker.

5. Conclusion

This paper presented a MILP model that optimizes the assignment of tasks and workers to stations in an assembly line, accounting for workers' learning and fatigue to minimize the makespan. The findings demonstrate that considering individual differences in workers' learning and fatigue can significantly reduce makespan compared to traditional balanced or random allocation strategies. In particular, assigning heavier workloads to workers with lower fatigue accumulation leads to more efficient task distribution and shorter production times.

The results also show that neglecting variability in learning and fatigue can cause substantial inefficiencies, with makespan increases of up to 28%. This underscores the importance of integrating both learning-aware and fatigue-sensitive assignment strategies into assembly line balancing, especially in human-driven production environments.

Future research could extend the model by incorporating ergonomic constraints such as task difficulty, physical limitations, and the need for rest and recovery. Additionally, introducing stochastic elements to capture uncertainty in learning and fatigue progression could improve the model's adaptability to real-world conditions. By addressing these aspects, future models can support the development of more sustainable, efficient, and worker-friendly assembly systems.

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