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Determining the Uncertainty of the Mean-Time-To-Failure via Nonlinear Regression Estimation

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Abstract

In industrial engineering reliability studies the Mean-Time-To-Failure (MTTF) is an accepted indication of the expected time to failure by the post-processing of the failure probability density function (PDF) data. Usually, the data of the failure PDF are fitted with Weibull, Lognormal or Gaussian distributions under assumptions of normality, with either chi-square or Maximum Likelihood Estimation (MLE) methods, however it is rarely the case that the uncertainty of the MTTF is also calculated due to the complexity of the analysis. In this paper the estimate of the uncertainty of the PDF parameters is formulated as a nonlinear regression problem and utilized as inputs into a corresponding measurement uncertainty equation for the MTTF that incorporates correlation effects in order to determine the nominal value and associated uncertainty of the MTTF for reported failure data of lithium-ion batteries with numerical experiments. Results demonstrate that the combination of the expected value and uncertainty for the MTTF can quantitatively estimate appropriate statistical lower and upper bounds for the MTTF to offer superior functionality and accuracy for reliability engineering applications.

Keywords
Mean-Time-To-Failure (MTTF), nonlinear regression, Uncertainty Quantification (UQ), covariance matrix, reliability analysis

1. Introduction

In reliability engineering studies for the failure analysis of components or systems, the Mean-Time-To-Failure (MTTF) is a common statistical measure to quantify the likely time for a component or system to fail. This is usually determined by first selecting and fitting an appropriate Probability Density Function (PDF) for known time and failure data in order to first quantify the estimates of the parameters of the failure PDF, and to then secondly incorporate the fitted PDF parameters when calculating the final value of the MTTF.

When fitting a model for the PDF an analyst may typically utilize either a least-squares chi-squared (χ^2) merit function or a Maximum Likelihood Estimate (MLE) approach. The qualitative difference between χ^2 and MLE approaches are that a χ^2 approach determines how well a particular PDF model fits the data where the focus is on determining the best type of model that aligns to the shape of the PDF, whilst a MLE approach is instead to appropriately estimate the model parameter values in such a way as to maximize the probability that the given model would be able to observe the given data i.e. that sampling from the fitted PDF is consistent with the original data.

As a result, when applying a regression analysis to determine the fitted model parameters of a particular PDF model, the model would always incorporate an intrinsic aleatoric uncertainty in the parameters that must also be logically propagated through the model when performing calculations with the PDF. Unfortunately, this is not often conventionally implemented in MTTF failure analysis calculations with the result that may then unintentionally lead to unforeseen potential statistical inconsistencies and interpretation errors. This research gap of current practice as reported in the literature is the motivation for the research study presented in this paper.

The research study focus of this paper is considered to be important as the uncertainty, or equivalently the tolerance or accuracy of the calculated MTTF, is crucial and fundamental reliability information that an analyst must always utilize in order to correctly interpret thresholds and limits for the corresponding lower and upper bounds for safe operating conditions. Under certain operating conditions the associated tolerance/uncertainty of the MTTF may potentially exceed the allowable nominal expected value of the MTTF based on manufacturing and/or production considerations that if not correctly accounted for, may unintentionally lead to the premature failure of the component/system to the detriment of operator safety and stakeholder interests.

To investigate and resolve this research gap, the problem statement is formulated as to conduct an investigation to determine an appropriate method to quantify the intrinsic uncertainty of the PDF model fitted parameters and to then incorporate this information in such a way as to propagate the uncertainty through the model to compute a logically consistent estimate of the associated uncertainty in the MTTF in order to specify lower and upper bounds for a confidence interval of the MTTF for reliability engineering practitioners.

1.1 Objectives

The research objectives of this paper are to firstly determine an appropriate method for a regression approach to accurately quantify the intrinsic uncertainty that may be attributed to the fitted PDF model parameters, and to secondly determine an appropriate method to propagate this uncertainty information through the PDF model to statistically determine a logically consistent confidence interval for the MTTF.

2. Literature Review

Typical probability density functions in reliability engineering include the Gaussian distribution of form $f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right]$ where t is the time, μ is the mean, and σ^2 is the variance, the conventional 2-parameter Weibull distribution $f(t;\lambda,k) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{t}{\lambda}\right)^k\right]$ where $k > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter and which is formally only defined for $t \geq 0$ and is zero for negative time, and the lognormal distribution that is defined as $f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left[-\frac{(ln(t)-\mu)^2}{2\sigma^2}\right]$ where μ and σ take on similar meanings as for the Gaussian distribution.

An extension to the 2-parameter Weibull distribution is the 3-parameter Weibull distribution that is defined by Safari et al (2025) in terms of a shape θ, scale β and location γ parameters as $f(t;\theta,\beta,\gamma) = \frac{\theta}{\beta}\left(\frac{t-\gamma}{\beta}\right)^{\theta-1} \exp\left[-\left(\frac{t-\gamma}{\beta}\right)^{\theta}\right]$ with the constraints that $t > \gamma,\ \theta > 0, \beta > 0$. For this particular distribution it has been proven that $MTTF(\theta,\beta,\gamma) = \gamma + \beta\,\Gamma\left(1+\frac{1}{\theta}\right)$ where θ models the aging characteristics of the system such that $\theta < 1$ indicates early life failures (negative ageing), $\theta = 1$ denotes that failures occur completely randomly (non-ageing failures), and $\theta > 1$ indicates wear out failures (positive ageing).

The MTTF is formally defined in terms of the failure probability density function $f(t)$ as $MTTF = \int_0^\infty t\,f(t)\ \mathrm{d}t = \int_0^\infty R(t)\ \mathrm{d}t$ where $R(t) = \exp\left[-\int_0^t h(x)\ \mathrm{d}x\right]$ is the reliability function, $F(t)$ is the unreliability function, and $h(t)$ is the hazard function as originally discussed by Martz (1987). The relationship between these functions is $h(t) = \frac{f(t)}{R(t)}$ and $R(t) = 1 - F(t)$ so that they may be calculated in terms of each other for mutual self-consistency. In principle the failure PDF $f(t)$ may take any appropriate form, however in the special case where the distribution is a Weibull the MTTF may then be analytically expressed as

$$MTTF = \lambda\Gamma\left(1+\frac{1}{k}\right) \qquad \text{Eq(1)}$$

where $\Gamma(z)$ is the Gamma function defined as $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\ \mathrm{d}t$.

Jordan (1978) analyzed the lognormal distribution in a modified form as $f(t)=\frac{1}{t\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}\left(ln\left(\frac{t}{t_m}\right)\right)^2\right]$ where the median life ML is defined as $ML=t_m=e^{\mu}$ and proved that the MTTF reduces to

$$MTTF=\bar{t}=t_m\exp\left(\frac{1}{2}\sigma^2\right) \qquad \text{Eq(2)}$$

Measurement theory has historically in engineering practice utilized the Kline-McClintock method however with the introduction of the Guide to the Uncertainty of Measurement (GUM) in 2020 by the BIPM et al (2020) uncertainty quantification for engineering measurements has evolved to also incorporate covariance contributions as investigated by Ramnath (2022). For a model of the form $y=f(x_1,\ldots,x_n)$ with uncertainties $u(x_1),\ldots,u(x_n)$ expressed as the square roots of variances with corresponding covariances $u(x_i,x_j),\ i,j\in[1,\ldots,n]$ the associated standard uncertainty is by the GUM formula of the form

$$u^2(y)=\sum_{i=1}^{n}\left(\frac{\partial y}{\partial x_i}\right)^2u^2(x_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}2\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j}cov(x_i,x_j) \qquad \text{Eq(3)}$$

The above formula in the special simplifying case of negligible correlations between the inputs reduces to the standard Kline-McClintock equation. The GUM uncertainty formula with and without covariance contributions follows a Bayesian statistics theoretical framework for mathematical consistency, where all of the inputs $x_i,\ i=1,\ldots,n$ are formally considered as random variables with associated variance/covariance uncertainties. This means that uncertainties in both natural variables such as the time as well as the uncertainties of fitted parameters $a_j, j=1,\ldots,M$ for a model equation $y=f(x_1,\ldots,x_n;a_1,\ldots,a_M)=f(x;a)$ may be incorporated for statistical consistency by simply treating the parameters themselves as new random variables regardless of the mathematical form of the model. In this approach there may be covariances between the original random variables $x_i,x_j\ i\neq j$ and covariances between the parameters $a_i,a_j,\ i\neq j$ however there will be zero covariances between the natural random variables and the artificial parameters i.e. $cov(x_i,a_j)=0,\ i=1,\ldots,n,\ j=1,\ldots,M$.

A practical consequence of this generality is that the GUM uncertainty framework may also in principle be applied to non-algebraic model equations such as two-layer Neural Networks (NNs) which approximate an algebraic model $y=f(x_1,\ldots,x_n)$ with a 2-layer (dual layer) NN model of the form $y=\hat{f}(x;\theta)=\frac{1}{N}\sum_{i=1}^{N}\sigma_\star(x;\theta_i)$. In this model equation N is the number of neurons which are typically optimized by trial and error, $\sigma_\star: R^d\times R^D\rightarrow R$ is a mapping function that is called the activation function where d is the dimension of x and D is a suitable integer, and $\theta_i=(\theta_1,\ldots,\theta_N)\in R^D$ is a hyper-parameter for the NN model. Different techniques exist for different choices of activation functions and determining the size of the dual layers, however regardless of the specific type/category of NN or equivalent AI model that is considered, the generality of the GUM uncertainty framework is still applicable from a mathematical modelling completeness and statistical consistency perspective.

As a result, the uncertainty using the GUM formula may always be propagated through the generalized model equation via the uncertainty variance/covariance information in the combination of the random variables x_i and parameters θ_i either deterministically if the model equation is analytically tractable, or stochastically if the model is not analytically tractable through for example Monte Carlo simulations. The equivalence in the GUM uncertainty framework for model equations between deterministic and stochastic approaches has been proven in earlier work by Van der Veen and Cox (2021).

Due to the complexity of the governing uncertainty equation for reliability engineering models as explored by Leszek and Bąk (2023) who could only analytically solve for the uncertainty in the MTTF for the simplified case of a hazardous function of the form $h(t)=at^b$ with the Kline-McClintock method and the challenges with computing the sensitivity coefficients, existing uncertainty estimate approaches have tended to focus on machine learning methods as discussed by Mubashir and Brantner (2025), active learning Kriging models for estimates of the prediction variance by Zhan et al (2024), and Markov chain approaches by Muscatello and Tolio (2024).

Artificial Intelligence (AI) uncertainty methods which evaluate an AI model's assessment of its own confidence of its predictions include traditional Monte Carlo Dropout (MCD) methods originally developed by Gal and Ghahramani (2016) that interprets the MCD as a Bayesian approximation. The MCD method may be applied to Neural Network

(NN) predictions to estimate epistemic uncertainty in a measurement model by repeated simulation events to quantify the aleatoric/random uncertainty by analyzing the data variability. A technical issue with determining the uncertainty in convolutional neural networks (CNNs) as a specialized category of more general NNs for applications such as image analysis feature identification and extraction, is that the corresponding optimization problem for obtaining the weights of the CNN is highly non-convex in a higher dimensional space so that there are always multiple different local minima for the corresponding loss function. Because the different optimization minima solutions are all technically equally valid, the different CNN models and their different weights therefore introduces an epistemic uncertainty component on top of the aleatoric uncertainty component from the intrinsic noise/uncertainty of the data as reported by Iagaru and Gottschling (2023). The MCD method has recently been implemented by Contreras and Bocklitz (2024) who adapted the original MCD method by incorporating a Gaussian Mixture Model (GMM) to determine the uncertainty in Raman optical spectra classifications by splitting the data into 'Certain' and 'Uncertain' groups in a convolutional neural network (CNN).

A competing method in AI uncertainty to the MCD method is deep ensembles which are composed of multiple independent NNs. These have been applied in deep learning studies and have been shown to outperform the standard MCD method for AI Uncertainty Quantification (UQ) for a variety of datasets when performing regressions i.e. fitting data with model parameters and in classifications of datasets by cloud computing service providers such AWS (2025), however a disadvantage of deep ensembles are that they are computationally expensive to implement and utilize. Refinements to deep ensemble methods include a post-hoc calibration method that was developed by Yang and Yee (2024) which was demonstrated to improve the AI uncertainty quantification in terms of the regression accuracy, the reliability of the estimated uncertainty, and the training efficiency.

A comprehensive survey on AI uncertainty in deep NNs by Gawlikowski et al (2023) outlines the different approaches such as deterministic NNs, Bayesian NNs (BNNs), ensembles of NNs, and test-time data augmentation approaches amongst others, for applications in fields such as medical image analysis, robotics, and earth observation studies, all of which require practical applications of AI uncertainties. A recently developed alternative known as the batch ensemble method by Thuy and Benoit (2025) is reported to match the deep ensemble method in both uncertainty and accuracy with computational cost savings in training time, test time and memory storage, for the particular application of industrial image classification.

AI reliability estimates methods qualitatively differ from AI uncertainty methods, as they assess instead the AI model's ability to perform its intended function consistently and correctly over time and under the same defined conditions for the particular purpose and application. An AI model that cannot quantify its uncertainty is by definition unreliable even if the fit is highly accurate. In the particular context of the topic studied in this paper the AI reliability estimate may simply be defined as the ability of the AI model to fit the failure PDF for a variety of supplied measurement data, and in such a way as to be reproducible as a proxy for consistency, numerically accurate as a proxy for correctness, and repeatable as a proxy for the same defined conditions of the intended purpose and application, of fitting the model parameters and parameter uncertainties of the failure PDF. These AI reliability metrics have been considered by Mortaji and Sadeghi (2024) who outlined the dimensions for AI reliability to encompass (i) consistency and predictive accuracy, (ii) robustness and adaptability, and finally (iii) transparency and explainability. The problem of AI reliability estimates has been approached from a reliability engineering and risk management perspective by Zhang et al (2025) and by a Design for Reliability (DfR) methodology by Yuan et al (2024).

Particular DfR tools as discussed by Werner and Schumeg (2024) include Failure Modes and Effects Analysis (FMEA) as studied by Li and Chignell (2022) as a safety engineering tool, Reliability Block Diagram (RBD) as studied by Janardhanan et al (2025) who extended the concept of a RBD to source/destination large topology networks which may be considered as representative of AI based NNs, Highly Accelerated Life Testing (HALT) and Physics of Failure Analysis (PoF), where Werner and Schumeg (2024) recommended that traditional DfR tools may with appropriate modifications also be usefully applied to the design of AI systems and consequently also be used to test for AI reliability estimates.

3. Methods

The methods for determining the uncertainty of the MTTF consists of combining the GUM uncertainty formula with inputs specified by the regression analysis estimates of the variances/covariances by either using the R statistical software package fitdistrplus that was developed by Delignette-Muller and Dutang (2015), the Python package statsmodels that was developed by Seabold and Perktold (2010), or the built in Matlab nonlinear regression model developed by the Mathworks (2025).

For brevity let the MTTF for the Weibull distribution be $\bar{t}_W = \lambda \cdot \Gamma\left(1 + \frac{1}{k}\right)$ so that by applying the GUM uncertainty formula it immediately follows that

$$u^2(\bar{t}_W) = \left[\frac{\partial \bar{t}_W}{\partial \lambda}\right]^2 u^2(\lambda) + \left[\frac{\partial \bar{t}_W}{\partial k}\right]^2 u^2(k) + 2\left[\frac{\partial \bar{t}_W}{\partial \lambda}\right] \cdot \left[\frac{\partial \bar{t}_W}{\partial k}\right] \cdot cov(\lambda, k) \qquad \text{Eq(4)}$$

In this formula $\frac{\partial \bar{t}_W}{\partial \lambda} = \Gamma\left(1 + \frac{1}{k}\right)$ may be explicitly calculated whilst $\frac{\partial \bar{t}_W}{\partial k}$ may instead for simplicity be approximated with finite differences as the analytical partial derivative of the Gamma function is mathematically expressed in terms of the digamma function as reported by Wolfram (2025).

Similarly for brevity for the MTTF of the lognormal distribution let $\bar{t}_L = e^{\mu + \frac{1}{2}\sigma^2}$ so that again by applying the GUM uncertainty formula as for the Weibull uncertainty, it immediately follows after various algebraic mathematical simplifications that

$$u^2(\bar{t}_L) = \left[e^{2\mu + \sigma^2}\right] \cdot \left[u^2(\mu) + \sigma^2 u^2(\sigma) + 2\sigma\, cov(\mu, \sigma)\right] \qquad \text{Eq(5)}$$

The general methodology of the algorithmic steps to synthesize/combine a nonlinear regression with a GUM uncertainty analysis to determine the tolerance/uncertainty in the MTTF is summarized as shown in the flowchart in Figure 1 for arbitrary failure PDF models.

The starting point is to load the failure PDF data which in general may be either Gaussian or possibly non-Gaussian so that the failure PDF may be of the form of a time t versus density of failure $f(t)$ graph. An example of non-Gaussian failure PDF is shown in Figure 2 based on an investigation by Pal et al (2025) who studied the failure of an injection molding machine for plastic soft drink bottles.

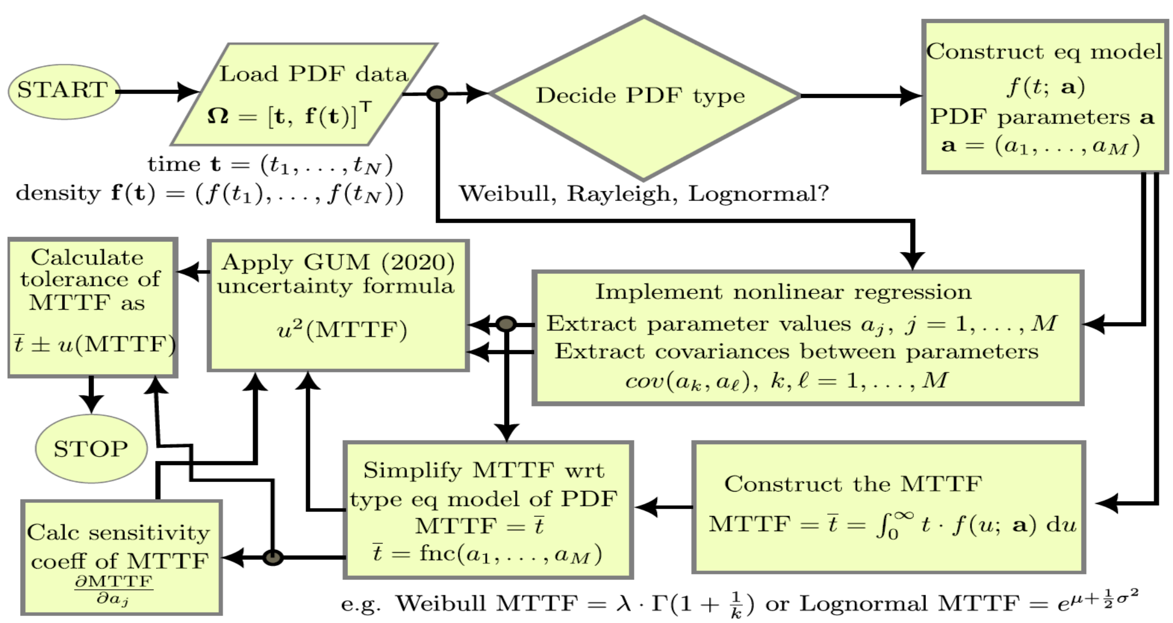

Figure 1. Flowchart of method to synthesize a nonlinear regression with a GUM uncertainty for the MTTF tolerance

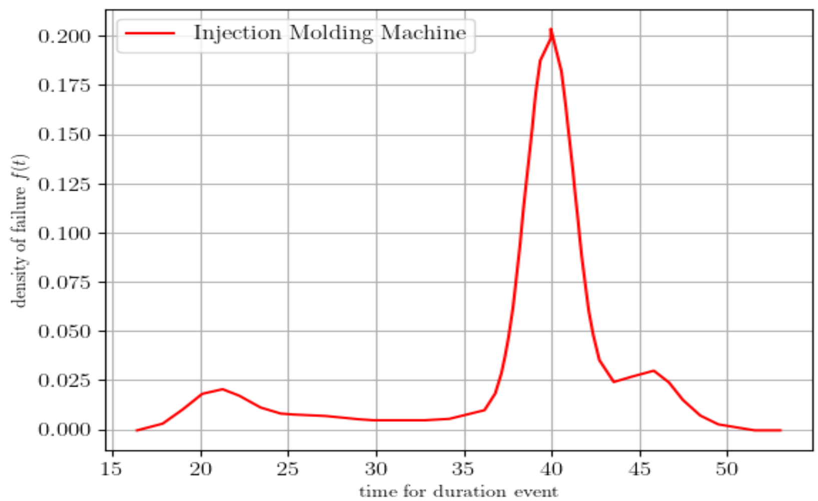

Figure 2. Non-Gaussian failure PDF data reported by Pal et al (2025) that shows asymmetries and multiple peaks

Once the failure PDF is loaded the next step is to decide on an appropriate model that can fit this data with any appropriate method such as engineering reliability judgement, theoretical insights, or numerical experimentation. In the context of failure analysis this may typically be Gaussian, Weibull, Rayleigh or Lognormal density distributions all of which are parametrized equations of the general form $f(t; a)$ as per the approach in the previous section where $a = (a_1, \dots, a_M)$. The underlying temporal/density data is then fitted with a nonlinear regression to determine the expected values of the parameters $a_j,\ j = 1, \dots, M$ and the associated covariances $cov(a_k, a_l),\ k, l \in [1, \dots, M]$ which completely models the failure PDF. The next step is to post-process the chosen MTTF by integrating out the time so that a formula for the MTTF may be expressed in terms of just the fitted parameters.

This may be achieved either by reference to the literature for known failure PDFs such as Weibull, Rayleigh or Lognormal distributions, or if necessary by analytical/symbolic integration to simplify the form of the MTTF such that it may be calculated entirely in terms of just the fitted nonlinear regression parameters. Once the form of the MTTF is expressed as an algebraic equation, the sensitivity coefficients are calculated, and the GUM uncertainty formula is applied to the MTTF to obtain the expected value of the MTTF and its associated standard uncertainty at the 1σ standard deviation level which corresponds to a 68% confidence interval. The expanded uncertainty at a 95% confidence interval is simply double the standard uncertainty, and this may be conveniently used to specify the MTTF uncertainty/tolerance limits.

4. Data Collection

Data for the failure statistics of commercial lithium-ion batteries were originally reported by Harris et al (2017) who performed experiments to determine the Remaining Useful Life (RUL) which was defined as the number of cycles as a proxy for the time until 80% of the remaining capacity of the battery was reached. The experimental data with a maximum of 593 cycles was analyzed using maximum likelihood techniques for the Gaussian, 2-parameter and 3-parameter Weibull failure probability density function $f(t)$. A sensitivity analysis of the original data by Harris et al (2017) was investigated by varying the failure criteria as 70%, 75%, 80%, 85% and 90% of the available battery capacity, and the authors concluded that a 3-parameter Weibull PDF gave inconsistent results as the Weibull shape parameter β changed too drastically.

The original dataset by Harris et al (2017) was subsequently analyzed by Mouais et al (2021) who considered Weibull, Lognormal and Gaussian PDFs and also investigated the influence of censored data and Lilliefors (modified Kolmogorov-Smirnov), Pearson's Chi-Squared and Cramer-von Mises goodness of fit tests on the failure PDF, from which they concluded that for this particular dataset that the lognormal distribution for the failure PDF is the best overall fit with the competing distributions as summarized in Table 1.

Table 1. Summary of MLE parameters for battery failure PDF reported by Mouais et al (2021)

Weibull	$\alpha = 5.22,\ \beta = 516.88$	$f(t) = 0.01\left(\frac{t}{516.88}\right)^{4.22} exp\left[-\left(\frac{t}{516.88}\right)^{5.22}\right]$
Lognormal	$\theta = 6.13,\ \tau = 0.28$	$f(t) = \frac{1}{0.28\sqrt{2\pi}t} exp\left[-\frac{1}{2}\left(\frac{ln(t) - 6.13}{0.28}\right)^2\right]$
Gaussian	$\mu = 470.4,\ \sigma = 119.32$	$f(t) = \frac{1}{119.32\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{t - 470.4}{119.32}\right)^2\right]$

To utilize this publicly available reported dataset the lognormal distribution may be written as $f(t) = \frac{1}{\sigma t\sqrt{2\pi}}\exp\left[-\frac{(ln(t)-\mu)^2}{2\sigma^2}\right]$ with $\mu = 6.13 \pm 0.01$ and $\sigma = 0.28 \pm 0.01$. The possible rectangular errors due to the numerical resolution of ± 0.01 for the expected value μ and standard deviation σ may then be converted to an equivalent Gaussian uncertainty by dividing by $\sqrt{3}$ and added to the nominal values of μ and σ to artificially simulate sampled values as reported by the JCGM GUM-6 standard in BIPM et al (2020), with an additional Gaussian noise of ± 0.0005 that may be added to the failure PDF to produce the representative synthetic data shown in Figure 3 for subsequent analysis in order to demonstrate the methodology in this paper.

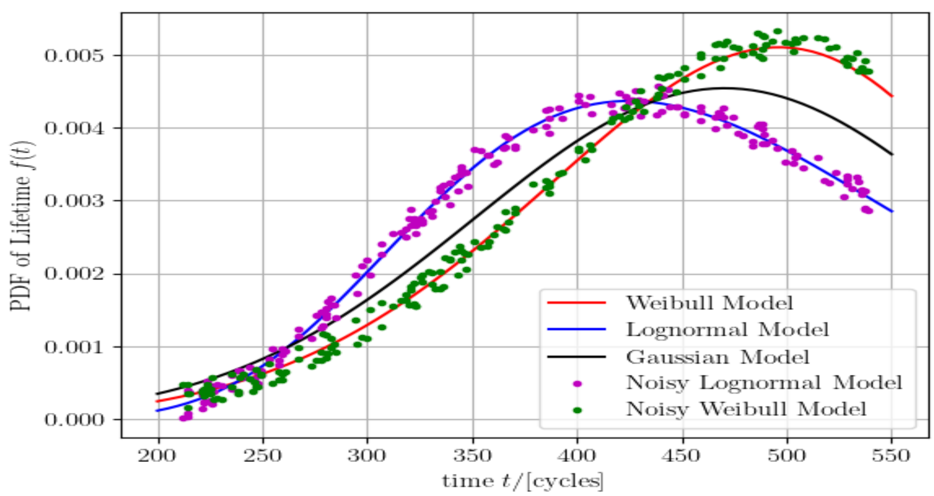

Figure 3. Synthetic battery data generated by applying noise to a representative Lognormal and Weibull failure PDF

Data for a different case study for the failure PDF of electrical transformers were originally reported by Chaniago et al (2024) for the time and Cumulative Distribution Function (CDF) as shown in Table 2. Recalling that $MTTF = \alpha \cdot \Gamma\left(1 + \frac{1}{\beta}\right)$ it may be shown that the CDF is of the equivalent form $F(t) = 1 - \exp\left[-\left(-\frac{t}{\alpha}\right)^{\beta}\right]$. In this special case by setting $x = \ln(t)$, $a = \beta$, and $b = -\beta\ln(\alpha)$, the original CDF may be transformed into the linear equation $y = ax + b$ and a linear regression applied to the transformed equation. The original analysis by Chaniago et al (2024) only determined the nominal values of the Weibull parameters and was not able to determine the corresponding parameter uncertainties in the Weibull distribution and hence the corresponding uncertainty in the MTTF.

Earlier work by Ramnath (2020) has developed MATLAB/GNU Octave computer codes for determining straight line parameter variances/covariances for experimentalists that may be applied to the special case of linearized models, however the nonlinear data may also instead be directly analyzed by recovering the failure PDF from the failure CDF using the fundamental statistical formula $f(t) = \frac{dF(t)}{dt}$ as shown in Figure 4 by using central differences to extract the gradient.

Table 2. Case study data for failure PDF of electrical transformers reported by Chaniago et al (2024)

t (days)	F(t)
75	0.0455
208	0.1104
210	0.1753
238	0.2403
265	0.3052
408	0.3701
476	0.4351
572	0.5000
608	0.5649
671	0.6299
744	0.6948
1266	0.7597
1525	0.8247
1532	0.8896
1607	0.9545

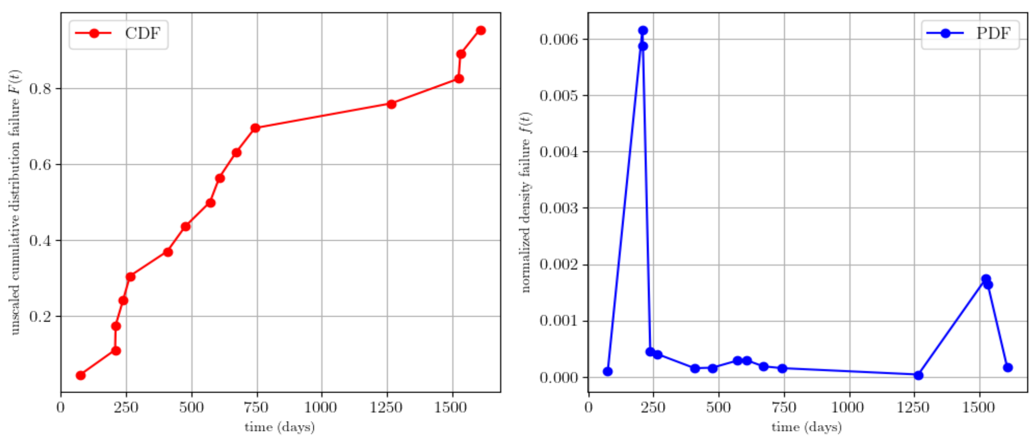

Figure 4. Case study data of electrical transformer failure by Chaniago et al (2024) to extract the PDF from the CDF

5. Results and Discussion
5.1 Numerical Results
Results for the nonlinear regression calculations of the lithium battery data were performed in Python 3.12 with the scipy.optimize curve_fit routine using the data of the Weibull and Lognormal distributions from Figure 3 and are summarized in Table 3.

The numerical results from Table 3 may then be substituted into Eq(1) and Eq(4) for the Weibull distribution, and into Eq(2) and Eq(5) for the Lognormal distribution, to determine the corresponding expected value and uncertainty TMMF for these distributions. When these formulae are substituted the final expressions that are yielded are then

$$u^2(\bar{t}_W) = \left[\Gamma\left(1+\frac{1}{k}\right)\right]^2 u^2(\lambda) + \left[\lambda\left(\frac{\Gamma\left(1+\frac{1}{k+\delta k}\right)-\Gamma\left(1+\frac{1}{k}\right)}{\delta k}\right)\right]^2 u^2(k) + 2\left[\Gamma\left(1+\frac{1}{k}\right)\right]\left[\frac{\Gamma\left(1+\frac{1}{k+\delta k}\right)-\Gamma\left(1+\frac{1}{k}\right)}{\delta k}\right] cov(\lambda, k) \quad \text{Eq(6)}$$

$$u^2(\bar{t}_L) = \left[e^{2\mu+\sigma^2}\right] \times \left[u^2(\mu) + \sigma^2 u^2(\sigma) + 2\sigma\, cov(\mu, \sigma)\right] \quad \text{Eq(7)}$$

Table 3. Summary of parameter variances and covariances for lithium battery case study

Weibull Distribution	$\boldsymbol{f(x\|\lambda,k) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}, \bar{t}_W = \lambda \cdot \Gamma\left(1+\frac{1}{k}\right)}$
Estimated scale parameter $\boldsymbol{\lambda} =$ $\mathbf{489.4095}$	Scale parameter $\boldsymbol{\lambda}$ c.i. [486.1775, 492.6416] at 95% significance
Estimated shape parameter $\boldsymbol{k} = \mathbf{6.6452}$	Shape parameter $\boldsymbol{k}$ c.i. [6.46211, 6.82833] at 95% significance Covariance matrix $C = \begin{bmatrix} 0.00861436 & -0.01719061 \\ -0.01719061 & 2.683824 \end{bmatrix}$
Lognormal Distribution	$\boldsymbol{f(x\|\mu,\sigma) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left\{\frac{-\ln t - \mu}{2\sigma^2}\right\}, \bar{t}_L = e^{\mu+\frac{1}{2}\sigma^2}}$
Estimated scale parameter $\boldsymbol{\mu} =$ $\mathbf{6.06843}$	Scale parameter $\boldsymbol{\mu}$ c.i. [6.06017, 6.076708] at 95% significance
Estimated shape parameter $\boldsymbol{\sigma} =$ $\mathbf{0.21864}$	Shape parameter $\boldsymbol{\sigma}$ c.i. [0.211677, 0.225612] at 95% significance Covariance matrix $C = \begin{bmatrix} 1.75662 \times 10^{-5} & 4.67324 \\ 4.67324 & 1.247283 \times 10^{-5} \end{bmatrix}$

5.2 Graphical Results

Substituting the nonlinear regression results from Table 3 into the final GUM uncertainty analysis formulae in Eq(6) and Eq(7) then produces the graphical results as shown in Figure 5 from which it may be observed that MTTF exhibits a natural variation around the expected value from which lower and upper tolerances may be specified. A similar analysis approach may also be applied to the electrical transformer case study data reported in Figure 4 which produces the results shown in Figure 6.

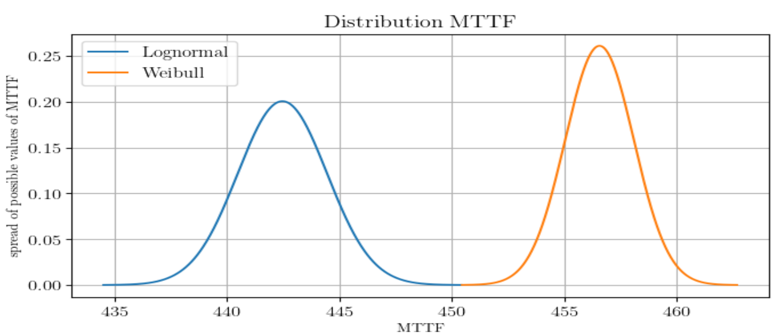

Figure 5. Results of computed TMMF values for Weibull and Lognormal failure PDF data for lithium battery data

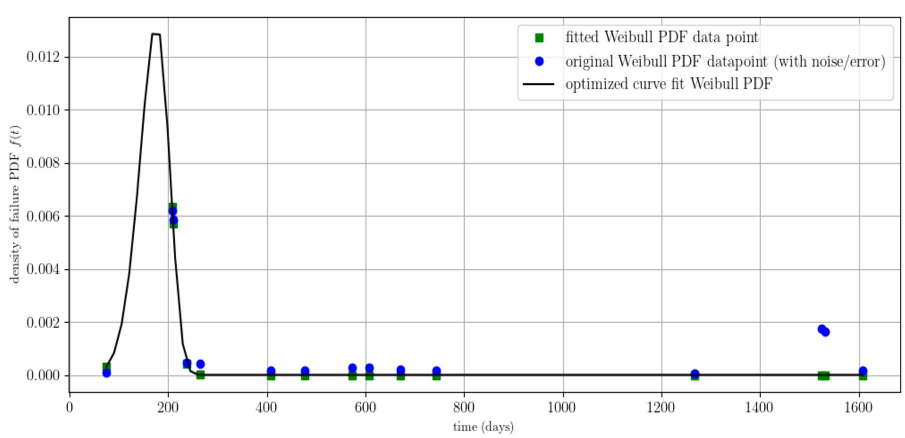

Figure 6. Results of computed Weibull PDF data for electrical transformer case data

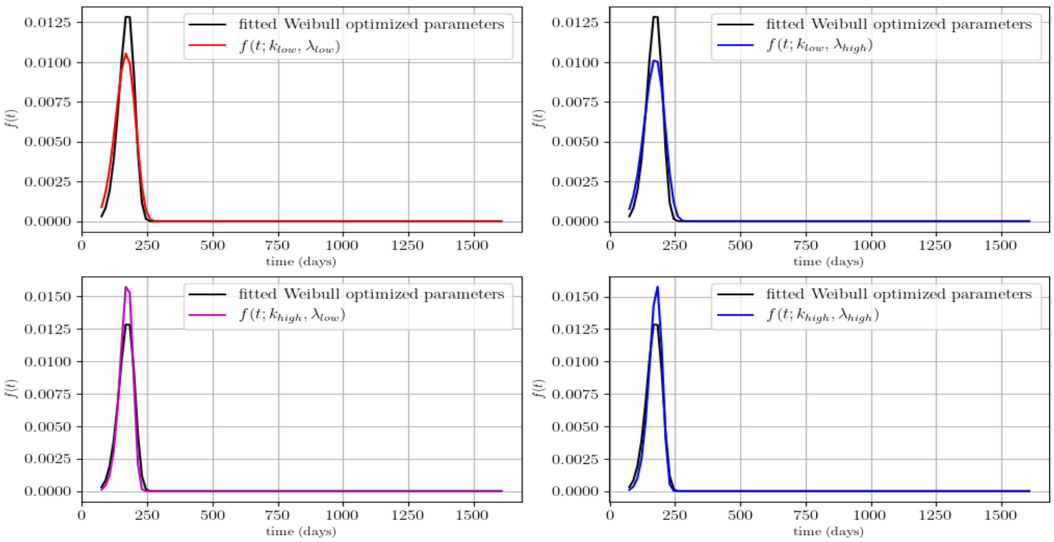

Figure 7. Results of variation of Weibull PDF parameter variations for electrical transformer case data

When the variations in the spread of the parameters obtained from the nonlinear regression for the Weibull PDF are also considered this then results in a set of curves that are visualized in Figure 7 that shows how the MTTF will vary based on the variation in the failure PDF curves, and will affect the MTTF tolerance/uncertainty.

5.3 Proposed Improvements

The method proposed in this paper has assumed a failure PDF that is either a Weibull or a Lognormal distribution which are 2-parameter densities, however the method is sufficiently general to work for any suitable PDF that can analytically model the failure of components or systems. A potential future improvement is to study the utility of 4-parameter density functions such as the generalized lambda distribution which may potentially offer improvements in refining the shape of the PDF particularly for skew-symmetric density distributions.

5.4 Validation

The governing equations in this paper were derived by hand for the distributions that were considered however it is possible to utilize symbolic algebra to validate the equations for future work that involves more algebraically complex distributions. This can be achieved with the aid of the Python package sympy as illustrated in Figure 8.

6. Conclusion

The research objectives of this paper were firstly to determine an appropriate method for a regression approach to accurately quantify the intrinsic uncertainty that may be attributed to the fitted PDF model parameters. This was achieved by implementing a nonlinear regression of the sampled data which incorporated aleatoric uncertainty contributions that were superimposed onto the sampled data with the underlying combination of resolution and uncertainty errors. The second research objective was determine an appropriate method to propagate this uncertainty information through the PDF model to statistically determine a logically consistent confidence interval for the MTTF, and this was achieved by implementing an extension of the Kline-McClintock method based on the GUM for model equations that incorporated the correlation information that was generated by the nonlinear regression analysis. As a result a coherent mathematical model was developed that can simultaneously combine the aleatoric uncertainty contribution effects from real world failure probability density function data into a consistent method that can propagate this information through the model to accurately predict accuracy and tolerance specifications for the MTTF. This is a new and unique research outcome in the application of failure analysis predictions that has not been previously reported and which avoids costly Monte Carlo simulations that were previously the only feasible approach to quantify the uncertainty in the predicted MTTF.

```python
# SYMBOLIC VALIDATION
mu_sym, sigma_sym, cov_musigma, u_mu, u_sigma = sp.symbols('mu sigma c delta_mu delta_sigma')
tbar_sym = exp(mu_sym + Rational(1, 2)*sigma_sym**2)
c_mu = diff(tbar_sym, mu_sym)
c_sigma = diff(tbar_sym, sigma_sym)
u_tbar = sqrt(  (c_mu*u_mu)**2 + (c_sigma*u_sigma)**2 + 2*c_mu*c_sigma*cov_musigma )
```

(a) Illustration of how symbolic variables are generated and processed

```
In [243]: u_tbar
Out[243]:
```

$$\sqrt{2c\sigma e^{2\mu+\sigma^2} + \delta_{\mu}^2 e^{2\mu+\sigma^2} + \delta_{\sigma}^2 \sigma^2 e^{2\mu+\sigma^2}}$$

```
In [244]:
```

(b) Illustration of how symbolic expressions are processed and rendered

Figure 8. Demonstration of how to utilize symbolic algebra functionality for extensions to other distributions

The method that has been developed that synthesizes a nonlinear regression and a GUM uncertainty analysis to work out the expected value and uncertainty of the MTTF may be computationally implemented in 1.1155 seconds on a modern laptop with a 1.10 GHz CPU and 8 GB of RAM running under Windows 11 64-bit. A typical implementation for the computer code may be scripted in roughly 178 lines of Python code without lower-level C/C++ compiled code numerical calls by taking advantage of the numpy and sympy Python libraries.

The main computational bottleneck is in determining the starting search values for the nonlinear regression analysis that must be achieved by trial and error for the initial nominal data. By contrast, if the same calculations were performed in a Monte Carlo simulation there would typically be a need for anymore from 15000 to 25000 Monte Carlo simulation events, therefore the computational time would be increased to at least $120\ \text{s} \times 15000 = 4.64\ \text{hr}$ by disregarding the additional post-processing of the generated Monte Carlo simulation data. This additional time of between half to a full day makes a Monte Carlo simulation impractical in production and manufacturing environments when real-time MTTF values/tolerances must be obtained.

The underlying method that has been developed will work for any analytical model, either algebraic or neural network based, that may be partially differentiated either analytically or symbolically, where the model PDF has parameters that may be specified with expected values and associated variances/covariances.

As a result, the method developed in this paper will work for a majority of reliability engineering and failure analysis PDFs either Gaussian or non-Gaussian that are reported in the literature. Limitations are that the method will not work for failure PDF models that cannot be analytically constructed or which exhibit multiple peaks.

An example of such a limitation is in the non-Gaussian data reported in Figure 2 which exhibits multiple peaks which cannot be easily modelled even with a sequence of piece-wise connected skew-symmetric Gaussian PDFs. Under these circumstances this failure PDF can instead only be numerically modelled either in terms of a kernel density estimation (KDE) or alternately an empirical distribution function (EDF), which must then be used to directly implement a direct numerical integration for the MTTF to compute its expected value and which be repeated in a Monte Carlo simulation to computationally determine the corresponding uncertainty/tolerance of the MTTF.

The method that has been developed will assist reliability engineering and safety/quality officers to determine the uncertainty/tolerance specifications for the MTTF in components and/or systems that can have critical and detrimental effects if not accurately quantified.

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Biography

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